MODULE 2

MATHEMATICS 30

Trigonometric and Circular Functions

Distance Learning

Alberta EDUCATION

Mathematics 30

Module 2

TRIGONOMETRIC AND CIRCULAR FUNCTIONS

Distance Learning

Alberta EDUCATION

This document is intended for	
Students	✓
Teachers (Mathematics 30)	✓
Administrators	
Parents	
General Public	
Other	

Mathematics 30
Student Module Booklet
Module 2
Trigonometric and Circular Functions
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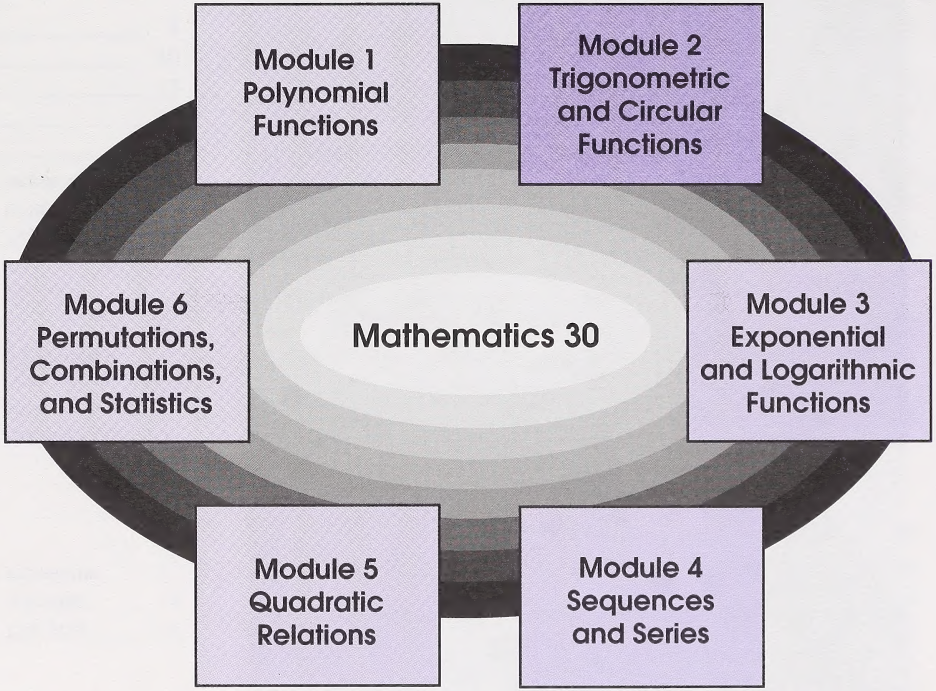
Welcome

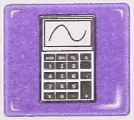
WESTFILE INC.

Welcome to Module 2. We hope you'll enjoy your study of Trigonometric and Circular Functions.

Mathematics 30 contains six modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.



The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

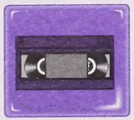


- Use your graphing calculator.

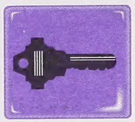
- Use your scientific calculator.

- Use computer software.

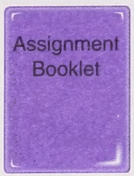
- Use the suggested answers in the Appendix to correct the activities.



- View a videocassette.

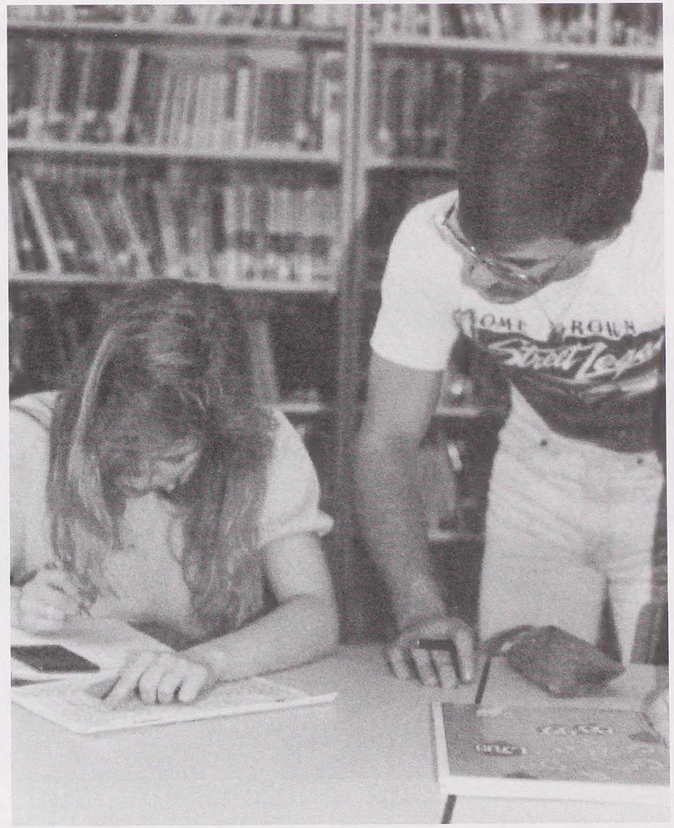


- Pay close attention to important words or ideas.



- Answer the questions in the Assignment Booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

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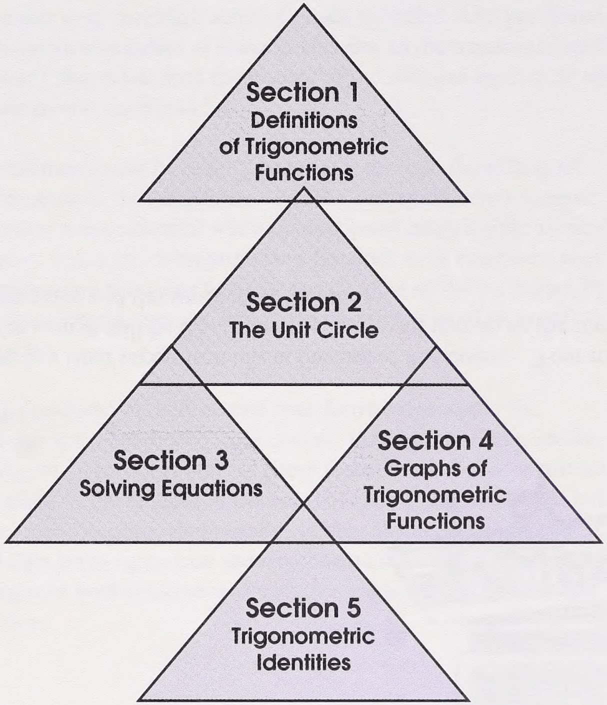
Module Overview

From ancient times navigators, surveyors, and builders have used triangles to solve measurement problems. The word *trigonometry* itself is derived from the Greek words *trigonon* and *metria*, which taken together mean **triangle measurement**. Today, the scope of trigonometry involves much more than triangle measurement; it encompasses sophisticated problems found in astronomy, physics, electronics, and engineering.

This module builds on your knowledge of right-triangle trigonometry. The basic trigonometric functions—sine, cosine, and tangent—and the functions formed from their reciprocals will be defined so that you can work with angles of any size, either positive or negative. These angles will be sketched in standard position in the coordinate plane. A special device called the *unit circle* will aid in determining the values of the trigonometric functions of these angles and assist in developing relationships among the functions. As a result you will be able to solve trigonometric equations, graph the trigonometric functions, and verify special relationships, called identities.

Module 2
Trigonometric and Circular Functions

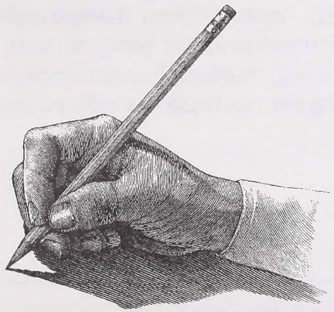


Evaluation

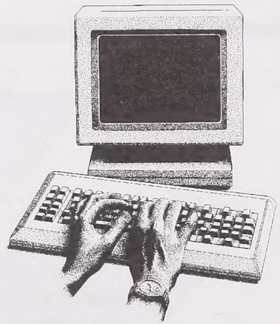
Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete five section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**13 marks**
Section 2 Assignment	**14 marks**
Section 3 Assignment	**13 marks**
Section 4 Assignment	**14 marks**
Section 5 Assignment	**16 marks**
Final Module Assignment	**30 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.



If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Definitions of Trigonometric Functions

Have you ever travelled across Canada by train? Did you know that the western expansion of Canada was tied to the transcontinental railway? But what does the history of the railroad have to do with trigonometric functions?

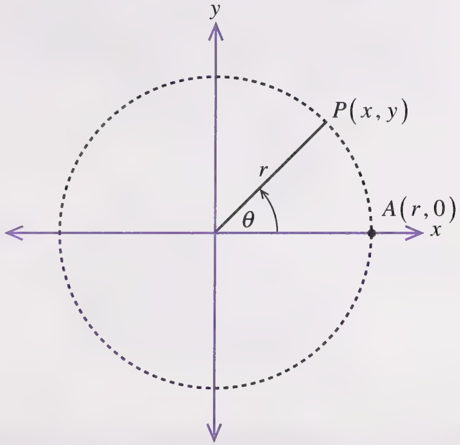
The railroad could have only been built through the efforts of surveyors and civil engineers. For the surveyors, trigonometric functions were essential when planning and mapping the route of the railway. Once the surveyors were finished, civil engineers used trigonometric functions to carry out the plan of the surveyors by building bridges, trusses, and so on to move around or avoid obstacles.

In this section you will extend your knowledge of the six trigonometric functions: sine, cosine, tangent, cosecant, secant, and cotangent. The definitions of these functions will not be restricted to the ratios of the lengths of the sides of the right triangle, but will be developed from the viewpoint of angles in standard position. You will then investigate how the trigonometric functions behave with changes in both positive and negative angles drawn in standard position.

Activity 1: Defining and Identifying Values of Trigonometric Ratios

An initial arm rotated through an angle θ to a terminal position is called the terminal arm. The terminal arm could fall anywhere in the coordinate system. You can determine a point P on the terminal arm whose coordinates are (x, y). You will define the distance from the origin to $P(x, y)$ as the **radius** $\boldsymbol{r}$. For any position of P in the coordinate plane, an angle θ is defined which represents the amount of rotation about the origin.

Start P at the point $A(r, 0)$ on the x-axis. As P moves through the coordinate system, it traces a path which is **circular**. The **radius** of this **circle** is equal to the length of the terminal arm r.

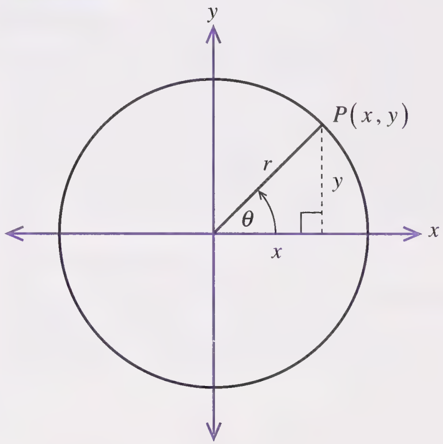


A **perpendicular** segment drawn from P to the x-axis will produce a right triangle. The Pythagorean Theorem can then be used to determine the length of the terminal arm r.

Note: The perpendicular is always drawn to the ***x*-axis**.

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}, \; r > 0$$



Since r is a length, it will always be a positive number.

Angle θ is measured from the positive x-axis.

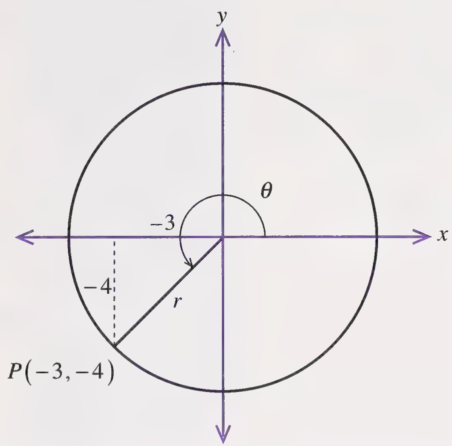
The six trigonometric ratios can now be defined in terms of x, y, and r.

- $\sin\theta = \frac{y}{r}$
- $\cos\theta = \frac{x}{r}$
- $\tan\theta = \frac{y}{x}$
- $\csc\theta = \frac{r}{y}$
- $\sec\theta = \frac{r}{x}$
- $\cot\theta = \frac{x}{y}$

P may be in any quadrant of the coordinate system. The following examples demonstrate this point.

Example 1

Given $P(-3,-4)$ on the terminal arm of an angle θ, find the exact values of the six trigonometric ratios.



Solution

Draw θ in standard position.

From the diagram, you can calculate r.

$$\begin{aligned} r^2 &= (-4)^2 + (-3)^2 \\ r^2 &= 16 + 9 \\ r^2 &= 25 \\ r &= 5 \end{aligned}$$

Although $r = \pm 5$, you choose $r = 5$ since $r > 0$.

$$\therefore \sin\theta = \frac{-4}{5} \qquad \cos\theta = \frac{-3}{5} \qquad \tan\theta = \frac{-4}{-3} = \frac{4}{3}$$

$$\csc\theta = \frac{5}{-4} \qquad \sec\theta = \frac{5}{-3} \qquad \cot\theta = \frac{-3}{-4} = \frac{3}{4}$$

In this example, $90° < \theta < 180°$, four of the six trigonometric ratios have negative values. Only the tan ratio and its reciprocal ratio, cotangent, are positive.

To test your understanding, try the next example on your own; then compare your answers with the solutions provided.

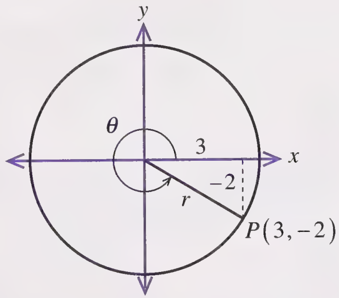
Example 2

The point $P(3,-2)$ is on the terminal arm of θ. Determine the values of the trigonometric ratios correct to four decimal places.

Solution

Draw θ in standard position.

From the diagram, calculate r.



$$r^2 = x^2 + y^2$$
$$r = \sqrt{3^2 + (-2)^2}$$
$$= \sqrt{13}$$

$\therefore \sin\theta = \dfrac{-2}{\sqrt{13}} \doteq -0.5547$ $\qquad \cos\theta = \dfrac{3}{\sqrt{13}} \doteq 0.8321$ $\qquad \tan\theta = \dfrac{-2}{3} \doteq -0.6667$

$\csc\theta = \dfrac{\sqrt{13}}{-2} \doteq -1.8028$ $\qquad \sec\theta = \dfrac{\sqrt{13}}{3} \doteq 1.2019$ $\qquad \cot\theta = \dfrac{3}{-2} = -1.5000$

Again, four of the six trigonometric ratios have negative values. In this case, only the cosine ratio and its reciprocal ratio, secant, are positive.

1. Given the following point on the terminal arm of angle θ in standard position, find the values of the six trigonometric ratios. Draw a diagram.

- **a.** $P(5,-12)$ (Use exact values.)
- **b.** $P(-1.5,1.25)$ (Answer to four decimal places.)
- **c.** $P(-1,-1)$ (Use exact values.)

2. Find the values of the six trigonometric ratios with the help of a diagram.

- **a.** $P(-3,-4)$ (Use exact values.)
- **b.** $M(-2.5,1.75)$ (Answer to four decimal places.)
- **c.** $T(7,-10)$ (Answer to two decimal places.)

3. A searchlight locates a plane in the sky at $P(-5,12)$. Find the exact values of the six trigonometric ratios.

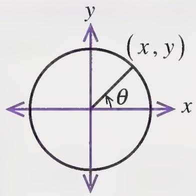
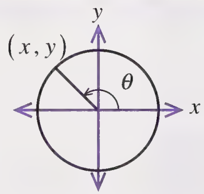
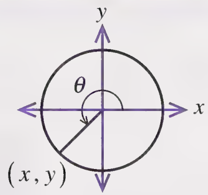
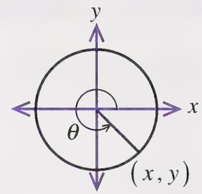
Check your answers by turning to the Appendix.

In Example 1, $P(-3,4)$ fell in the second quadrant. In this quadrant, x is **negative** and y and r are both positive. Those ratios involving x were **negative**.

In Example 2, $P(3,-2)$ fell in the fourth quadrant. Here, x and r are positive and y is **negative**. Those ratios involving y were **negative**.

It should be evident from these examples that the sign (+ or –) of the trigonometric ratio of θ is dependent upon the sign of x and/or y, as P rotates through the coordinate system.

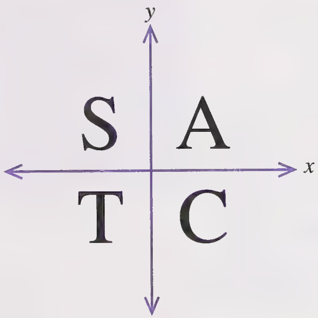
This information can be organized in a table. Consider a general point (x, y). Based upon the definitions of the six trigonometric ratios, the ratio is positive or negative in each of the four quadrants.

Trigonometric Ratio	First Quadrant $0° < \theta < 90°$	Second Quadrant $90° < \theta < 180°$	Third Quadrant $180° < \theta < 270°$	Fourth Quadrant $270° < \theta < 360°$
	 $x > 0,\ y > 0,\ r > 0$	 $x < 0,\ y > 0,\ r > 0$	 $x < 0,\ y < 0,\ r > 0$	 $x > 0,\ y < 0,\ r > 0$
$\sin\theta = \frac{y}{r}$	+	+	–	–
$\cos\theta = \frac{x}{r}$	+	–	–	+
$\tan\theta = \frac{y}{x}$	+	–	+	–
$\csc\theta = \frac{r}{y}$	+	+	–	–
$\sec\theta = \frac{r}{x}$	+	–	–	+
$\cot\theta = \frac{x}{y}$	+	–	+	–

The solutions are given in the following paragraph.

You should have discovered that, in the first quadrant, all the ratios are positive; in the second quadrant, only the sine ratio and the cosecant ratio (reciprocal ratio of sine) are positive; in the third quadrant, only the tangent ratio and the cotangent ratio (reciprocal ratio of tangent) are positive; and finally, in the fourth quadrant, only the cosine ratio and the secant ratio (reciprocal ratio of cosine) are positive.

These observations can be summarized by a simple acronym called the **CAST rule**. This rule identifies the **primary ratios that are positive** in each quadrant.



C – cosine
A – all
S – sine
T – tangent

R – rather
U – useful
L – little
E – expression

4. Use the CAST rule to determine the quadrant containing the terminal arm of θ for each of the following:

- **a.** $\sin\theta < 0$ and $\cos\theta > 0$
- **b.** $\tan\theta > 0$ and $\cos\theta > 0$
- **c.** $\csc\theta < 0$ and $\cot\theta > 0$
- **d.** $\sec\theta > 0$ and $\sin\theta < 0$
- **e.** $\sin\theta > 0$ and $\cos\theta < 0$

5. Use the CAST rule to determine the quadrant containing the terminal arm of θ for each of the following:

- **a.** $\cos\theta > 0$ and $\csc\theta < 0$
- **b.** $\cot\theta < 0$ and $\sin\theta > 0$
- **c.** $\sec\theta > 0$ and $\tan\theta > 0$
- **d.** $\csc\theta < 0$ and $\cot\theta < 0$
- **e.** $\sin\theta < 0$ and $\sec\theta < 0$

Check your answers by turning to the Appendix.

The following examples will test your ability to apply the CAST rule.

Example 3

In which quadrant(s) does the terminal arm of θ lie for $\cos\theta > 0$?

Solution

The terminal arm for $\cos\theta > 0$ lies in the first and fourth quadrants.

Example 4

In which quadrant(s) does the terminal arm of θ lie for $\csc\theta > 0$?

Solution

$\csc\theta = \frac{1}{\sin\theta}$

Since $\sin\theta > 0$ in the first and second quadrants, $\csc\theta$ must be positive in the first and second quadrants as well.

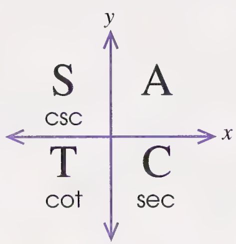
Example 5

In which quadrant(s) does the terminal arm of θ lie for $\tan\theta < 0$?

Solution

Since $\tan\theta > 0$ in the first and third quadrants, it must therefore be negative in the second and fourth quadrants.

Here are the reciprocals that are positive in each quadrant.



Example 6

In which quadrant(s) does the terminal arm of θ lie for $\sec\theta > 0$ and $\sin\theta < 0$?

Solution

Since $\sec\theta > 0$ in the first and fourth quadrants, and $\sin\theta < 0$ in the third and fourth quadrants, $\sec\theta > 0$ and $\sin\theta < 0$ in the fourth quadrant.

6. In which quadrant(s) does the terminal arm of θ lie for each of the following:

   a. $\sin\theta > 0$ **b.** $\sec\theta > 0$
   c. $\cos\theta < 0$ and $\csc\theta > 0$ **d.** $\tan\theta < 0$ and $\sec\theta > 0$

7. In which quadrant does the terminal arm of θ lie for each of the following:

   a. $\tan\theta < 0$ **b.** $\sec\theta > 0$
   c. $\sin\theta < 0$ and $\cot\theta > 0$ **d.** $\cos\theta > 0$ and $\sec\theta > 0$

Check your answers by turning to the Appendix.

In this activity you saw how the CAST rule works. The cofunctions are positive in each quadrant.

Activity 2: Five Values from One Value

When a trigonometric ratio is given, you can list the other five trigonometric ratios by applying the CAST rule for $0° < \theta < 360°$.

$0° < \theta < 360°$ is read as **θ greater than 0° and less than 360°**.

You will see this in the following examples.

Example 1

If $\cos\ \theta = \frac{2}{\sqrt{5}}$, list the exact values of the other trigonometric ratios for $0° < \theta < 360°$.

Solution

Since no specific quadrant has been identified, you must consider both quadrants where $\cos\ \theta > 0$. The CAST rule tells you that $\cos\ \theta$ is positive in the first and fourth quadrants.

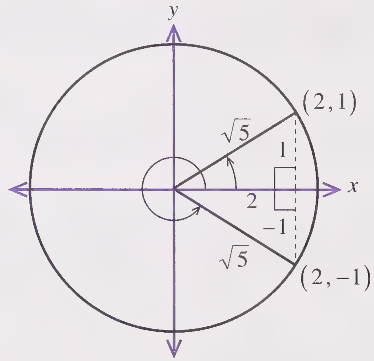
$$\cos\ \theta = \frac{x}{r}$$

One possibility is $x = 2$ and $r = \sqrt{5}$.

$$r^2 = x^2 + y^2$$
$$\left(\sqrt{5}\right)^2 = (2)^2 + y^2$$
$$5 = 4 + y^2$$
$$1 = y^2$$
$$y = \pm 1$$

Remember that every non-negative number has two square roots. Since y is a coordinate, it may be positive or negative. In the first quadrant, y is positive; in the fourth quadrant, y is negative.

You can now draw a diagram of a triangle with the given sides in the first and fourth quadrants and list the trigonometric ratios.



First Quadrant

$x = 2,\ y = 1,\ r = \sqrt{5}$

$\sin\ \theta = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$

$\cos\ \theta = \dfrac{2}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$

$\tan\ \theta = \dfrac{1}{2}$

$\csc\ \theta = \sqrt{5}$

$\sec\ \theta = \dfrac{\sqrt{5}}{2}$

$\cot\ \theta = 2$

Remember to rationalize the denominator whenever possible.

Fourth Quadrant

$x = 2,\ y = -1,\ r = \sqrt{5}$

$\sin\ \theta = \dfrac{-1}{\sqrt{5}} = -\dfrac{\sqrt{5}}{5}$

$\cos\ \theta = \dfrac{2\sqrt{5}}{5}$

$\tan\ \theta = \dfrac{-1}{2}$

$\csc\ \theta = -\sqrt{5}$

$\sec\ \theta = \dfrac{\sqrt{5}}{2}$

$\cot\ \theta = -2$

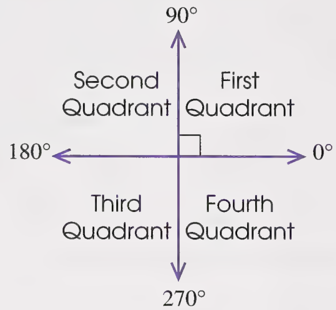
The CAST rule confirms the signs associated with these answers. In the first quadrant, all the ratios are positive; and in the fourth quadrant, only $\cos\ \theta$ and $\sec\ \theta$ (reciprocal ratio of $\cos\ \theta$) are positive.

The next example is similar to Example 1. Try to do it on your own before looking at the solution.

Example 2

If $\csc\ \theta = -1.25$ and $180° < \theta < 270°$, find the values of the other trigonometric functions correct to two decimal places.

Recall that $180° < \theta < 270°$ is read as **θ is greater than 180° and less than 270°**. This means that θ is in the third quadrant.



Solution

$\csc\ \theta = \frac{r}{y}$ and θ terminates in the third quadrant.

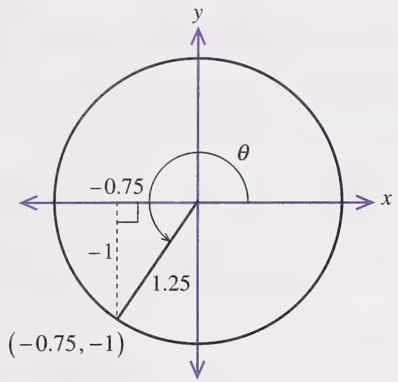
Thus, $\csc\ \theta = -1.25 = \frac{1.25}{-1}$.

Assume $r = 1.25$ and $y = -1$.

r is always positive.

$$\begin{aligned}(1.25)^2 &= x^2 + (-1)^2\\ x^2 &= (1.25)^2 - (-1)^2\\ x &= \pm\sqrt{(1.25)^2 - 1}\\ &= \pm 0.75\end{aligned}$$

In the third quadrant, $x = -0.75$.



$\therefore\ \sin\theta = \dfrac{-1}{1.25} = -0.80$ $\qquad \cos\theta = \dfrac{-0.75}{1.25} = -0.60$ $\qquad \sec\theta = \dfrac{1.25}{-0.75} = -1.67$

$\tan\theta = \dfrac{-1}{-0.75} = 1.33$ $\qquad \cot\theta = \dfrac{-0.75}{-1} = 0.75$

Again the CAST rule confirms the signs associated with the answers. In the third quadrant, only $\tan\theta$ and $\cot\theta$ (reciprocal ratio of $\tan\theta$) are positive.

1. If $\sin\theta = \frac{-\sqrt{3}}{2}$ and $180° < \theta < 270°$, determine the exact values of the other trigonometric ratios. Draw a diagram.

2. If $\sec\theta = \frac{-\sqrt{29}}{5}$ and θ terminates in the second quadrant, answer the following questions.

a. Sketch a diagram of the angle.
b. Find the exact values of the other trigonometric ratios.

3. If $\tan\theta = 1.0000$, list the values of the other trigonometric ratios for $0° < \theta < 360°$. Express your answer to four decimal places. (Check all quadrants!)

4. If $\cos\theta = \frac{-\sqrt{5}}{3}$ and $90° < \theta < 180°$, determine the exact values of the other trigonometric ratios. Draw a diagram.

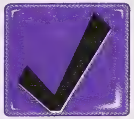
5. If $\sec\theta = -1.5$, list the values of the other trigonometric ratios for $0° < \theta < 270°$. Express your answer to four decimal places.

6. If $3\cot\theta = -\sqrt{3}$ and θ is in the fourth quadrant, find $\sin\theta$. Express your answer to two decimal places.

7. If $\cos\theta = \frac{-2}{3}$ and $\tan\theta$ is positive, find $\csc\theta$. Write the answer in exact values.

8. If $\tan \theta = \frac{-4}{3}$ and θ terminates in the fourth quadrant, answer the following questions.

a. Sketch a diagram of the angle.
b. Find the exact values of the other trigonometric ratios.

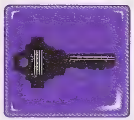


Check your answers by turning to the Appendix.

You can now find the value of each of the other trigonometric ratios when the value of one ratio is known.

Activity 3: Determining Reference Angles

So far you have looked at the values of the trigonometric ratios in relationship to a point on the terminal arm of angle θ. You will now extend this concept to consider the values of the trigonometric ratios of specific angles, including **obtuse angles**. It is necessary, however, to define a new term:

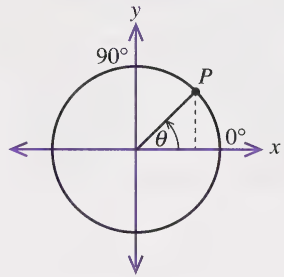


A **reference angle** is the angle formed between the terminal arm of a given angle and the x-axis. The reference angle is an **acute angle**.

Reference angles are always positive.

In general, any angle θ between 0° and 360° can be related to an acute angle R. Angle R is the angle between the terminal side of θ and the x-axis. The following diagrams help define the reference angle R for angles terminating in the four different quadrants.

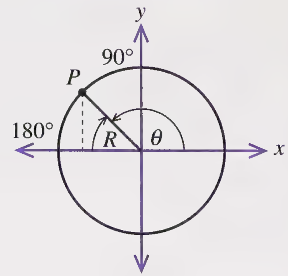
First Quadrant



$0° < \theta < 90°$

$R = \theta$

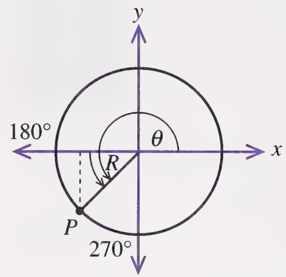
Second Quadrant



$90° < \theta < 180°$

$R = 180° - \theta$

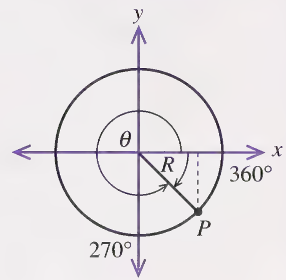
Third Quadrant



$180° < \theta < 270°$

$R = \theta - 180°$

Fourth Quadrant

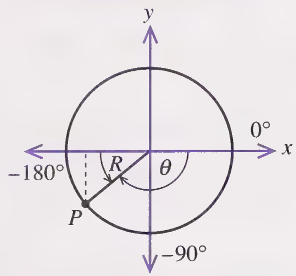


$270° < \theta < 360°$

$R = 360° - \theta$

If the rotation is clockwise, then θ is a negative angle.

Negative Angle



The reference angle lies in the third quadrant because rotation of the original angle is clockwise between $-90°$ and $-180°$.

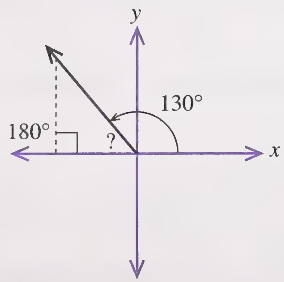
$$R = 180° + \theta$$

Look at some examples that demonstrate how to find reference angles.

Example 1

State the reference angle for 130°.

Solution



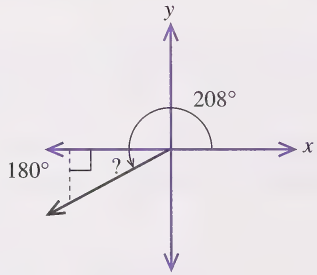
An angle of 130° is in the second quadrant. Therefore, subtract 130° from 180°. (Refer to the preceding second-quadrant diagram.)

$$\begin{aligned} R &= 180° - 130° \\ &= 50° \end{aligned}$$

Example 2

State the reference angle for 208°.

Solution



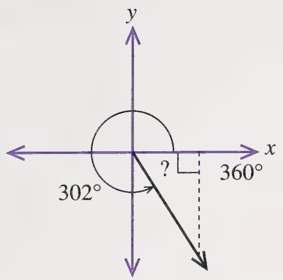
An angle of 208° is in the third quadrant. Therefore, subtract 180° from 208°. (Refer to the third-quadrant diagram shown earlier in this activity.)

$$\begin{aligned} R &= 208° - 180° \\ &= 28° \end{aligned}$$

Example 3

State the reference angle for 302°.

Solution



An angle of 302° is in the fourth quadrant. Therefore, subtract 302° from 360°. (Refer to the fourth-quadrant diagram shown earlier in this activity.)

$$R = 360° - 302°$$
$$= 58°$$

1. State the quadrant in which each angle terminates, and state the measure of the reference angle.

a. 169°	**b.** 343°	**c.** 486°	**d.** 256°
e. −197°	**f.** 155°	**g.** −120°	**h.** 525°
i. 246°	**j.** −212°		

Check your answers by turning to the Appendix.

Why are these reference angles so important? Recall that the trigonometric ratios were defined for **acute** angles. Hence, refer to the reference angle to determine the magnitude of the trigonometric ratios of an **obtuse** angle. Using the CAST rule, you know the signs associated with these ratios on the coordinate system.

To demonstrate this relationship, consider Example 3.

The reference angle of 58° in the right triangle would establish the magnitude of the trigonometric ratios. Using the CAST rule, only the cosine ratio and its recriprocal ratio (secant) would be positive.

$$\therefore \ \sin 302° = -\sin 58°$$
$$\doteq -0.8480$$

Refer to the reference angle to obtain the magnitude of the trigonometric ratio.

Use a calculator to verify this approximation.

3 0 2 sin

-0.848048096

Therefore, $\sin 302° \doteq -0.8480$.

Can you predict the relation between the angle 302° and the reference angle 58° for the other trigonometric ratios?

$$\cos 302° = \cos 58°$$
$$\tan 302° = -\tan 58°$$
$$\csc 302° = -\csc 58°$$
$$\sec 302° = \sec 58°$$
$$\cot 302° = -\cot 58°$$

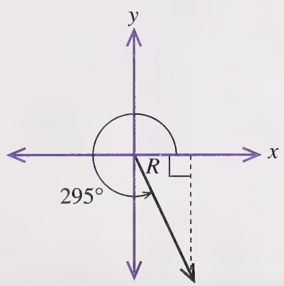
Example 4

State the relationship for the trigonometric function of $\cos 295°$ as related to the corresponding trigonometric function of its reference angle.

Solution

Follow these steps.

Step 1: Sketch the angle.



Step 2: Determine the reference angle.

$$\begin{aligned} R &= 360° - 295° \\ &= 65° \end{aligned}$$

Step 3: Use the CAST rule to determine the sign of the desired ratio.

In the fourth quadrant, $\cos \theta$ is positive.

Step 4: State the value of the ratio.

$$\cos 295° = \cos 65°$$

Example 5

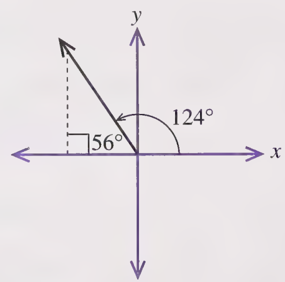
State the relationship for the trigonometric function of $\cot 124°$ as related to the corresponding trigonometric function of its reference angle.

Solution

$$\begin{aligned} R &= 180° - 124° \\ &= 56° \end{aligned}$$

In the second quadrant, $\tan \theta$ and $\cot \theta$ are negative.

$\therefore \cot 124° = -\cot 56°$

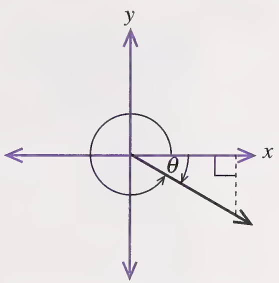


To apply the CAST rule, recall that $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, and $\tan\theta = \frac{1}{\cot\theta}$.

2. Place + or − in each blank.

a. $\sin 178° =$ ____ $\sin 2°$
b. $\cos 415° =$ ____ $\cos 55°$
c. $\sec 193° =$ ____ $\sec 13°$
d. $\tan 545° =$ ____ $\tan 185°$
e. $\sin 288° =$ ____ $\sin 72°$
f. $\csc 125° =$ ____ $\csc 55°$
g. $\tan 212° =$ ____ $\tan 32°$
h. $\cos(-190°) =$ ____ $\cos(-10°)$

3. In the fourth quadrant, the general form of an angle is expressed in one of two ways, $(360° - \theta)$ or $(-\theta)$, where $0° < \theta < 90°$.



$R = \theta$

The positive angle is $(360° - \theta)$.

The negative angle is $(-\theta)$.

Place + or − in each blank to make a true statement.

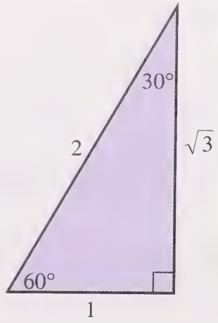
a. $\sin(360° - \theta) =$ ____ $\sin\theta$
b. $\cos(360° - \theta) =$ ____ $\cos\theta$
c. $\tan(360° - \theta) =$ ____ $\tan\theta$
d. $\csc(360° - \theta) =$ ____ $\csc\theta$
e. $\sec(360° - \theta) =$ ____ $\sec\theta$
f. $\cot(360° - \theta) =$ ____ $\cot\theta$

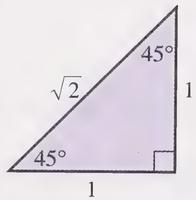
4. If $0° < \theta < 90°$, insert the correct sign, + or −, for each to make a true statement.

a. $\sin(360° - \theta) =$ ____ $\sin\theta$
b. $\tan(180° + \theta) =$ ____ $\tan\theta$
c. $\csc(180° - \theta) =$ ____ $\csc\theta$
d. $\cot(360° - \theta) =$ ____ $\cot\theta$
e. $\sec(180° + \theta) =$ ____ $\sec\theta$
f. $\cos(360° - \theta) =$ ____ $\cos\theta$

Check your answers by turning to the Appendix.

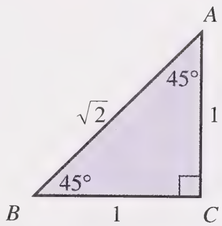
Before trying Examples 6 and 7, which illustrate how angles and reference angles are related, recall the following triangles from previous studies of trigonometry. These are called **special triangles**. Study the following special triangles.





You can find the exact trigonometric ratios of 45° using a 45°–45°–90° triangle.

ABC is a right triangle with $\angle A = \angle B = 45°$. Since $\angle A = \angle B$, the triangle is isosceles and $AC = BC$. Each of these equal sides is given a relative length of 1 unit.



Using the Pythagorean Theorem, AB is calculated as follows:

$$\begin{aligned}(AB)^2 &= 1^2 + 1^2 \\ AB &= \sqrt{2}\end{aligned}$$

The exact trigonometric ratios are as follows:

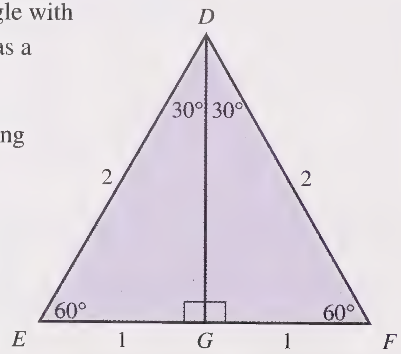
- $\sin 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
- $\cos 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
- $\tan 45° = \dfrac{1}{1} = 1$
- $\csc 45° = \dfrac{\sqrt{2}}{1} = \sqrt{2}$
- $\sec 45° = \dfrac{\sqrt{2}}{1} = \sqrt{2}$
- $\cot 45° = \dfrac{1}{1} = 1$

You will now find the exact trigonometric ratios of 30° and 60° from the next triangle.

ΔDEF is an equilateral triangle with $DE = EF = FD$. Each side has a relative length of 2 units.

The length of DG is found using the Pythagorean Theorem.

$$\begin{aligned}(DE)^2 &= (DG)^2 + (EG)^2 \\ (2)^2 &= (DG)^2 + (1)^2 \\ 4 - 1 &= (DG)^2 \\ \sqrt{3} &= DG\end{aligned}$$



The exact trigonometric ratios of 30° are as follows:

$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc 30° = \frac{2}{1} = 2 \qquad \sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \cot 30° = \frac{\sqrt{3}}{1} = \sqrt{3}$$

The exact trigonometric ratios of 60° are as follows:

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \frac{\sqrt{3}}{1} = \sqrt{3}$$

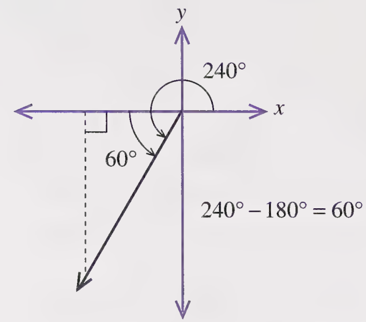
$$\csc 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \sec 60° = \frac{2}{1} = 2 \qquad \cot 60° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Example 6

Determine the exact value of $\csc 240°$.

Solution

In the third quadrant, $\csc \theta < 0$.



$\therefore \csc 240° = -\csc 60°$

Use the values from the 30°–60°–90° triangle.

$$\csc 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Thus, $\csc 240° = \frac{-2\sqrt{3}}{3}$.

Example 7

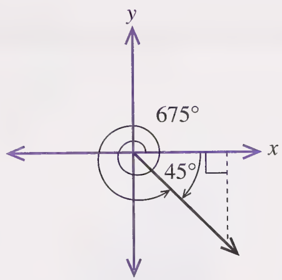
Determine the exact value of $\tan 675°$.

Solution

$$R = 720° - 675° = 45°$$

In the fourth quadrant, $\tan\ \theta < 0$.
Therefore, $\tan\ 675° = -\tan\ 45°$.

From the 45°–45°–90° triangle,
$\tan\ 45° = 1$. Therefore, $\tan\ 675° = -1$.



There are special rules to find the reference angles for angles greater than 360° or less than 0°.

To define the trigonometric functions for a given angle greater than 360° or less than 0° (a negative angle), you must first determine the positive angle between 0° and 360° that is coterminal with the given angle.

The trigonometric functions depend only on the position of the terminal arm. Hence, two **coterminal angles** will have the same trigonometric values.

The following example illustrates this.

Example 8

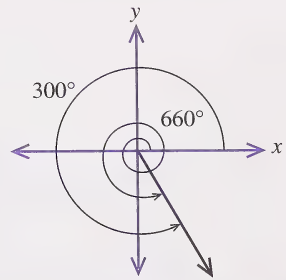
Find the positive angle between 0° and 360° that is coterminal with an angle of 660°.

Solution

First, subtract multiples of 360° from 660° until a positive angle between 0° and 360° is determined.

For example, $660° - 360° = 300°$.

Thus, 300° is coterminal with 660°. Examine the diagram.



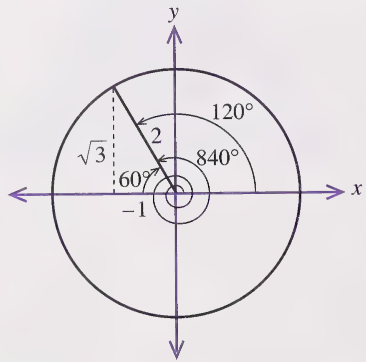
In the following examples, you will find the reference angles.

Example 9

Find the exact value of $\sec\ 840°$ using reference angles.

Solution

First, find the positive angle between 0° and 360° that is coterminal with 840°. To obtain this coterminal angle, subtract multiples of 360° from the given angle until the value is less than 360°.



$$\text{Thus, } 840° - 2(360°) = 840° - 720° \\ = 120°$$

Therefore, 120° is coterminal with 840°.

Now determine the reference angle for 120°.

$$R = 180° - 120° \\ = 60°$$

$$\therefore \ \sec 840° = \sec 120° \\ = -\sec 60° \\ = -2$$

In the second quadrant, secant is negative.

Use a calculator to check your answer.

8 4 0 cos INV $\frac{1}{x}$

-2.

Example 10

Give the exact value of $\sin(-135°)$ by finding the reference angle.

Remember that the reference angle is the positive acute angle between the x-axis and the terminal side of θ.

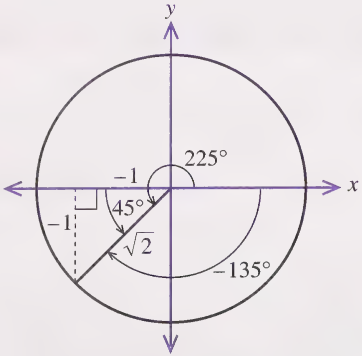
Solution

The reference angle for $-135°$ is 45°. (Refer to the diagram.) If you use the formula $(\theta - 180°)$ to obtain the reference angle, it will not work. If you want to use the formula, then you will have to find the positive value of θ that is coterminal with $-135°$. You can add 360° to $-135°$ in order to find an angle between 0° and 360° that is coterminal with $-135°$. Thus, $360° - 135° = 225°$ is coterminal with $-135°$. You can now use the formula to determine the reference angle.

$$R = 225° - 180° \\ = 45°$$

$$\therefore \ \sin(-135°) = -\sin 45° \\ = -\frac{1}{\sqrt{2}} \\ = -\frac{\sqrt{2}}{2}$$

sine is negative in the third quadrant.



As you can see by the examples, the angle θ places the terminal arm in a particular quadrant. A right triangle is formed by drawing a perpendicular from a point on the terminal arm **to the *x*-axis**. The trigonometric ratios for θ are then determined by referring to the reference angle between the terminal arm and the x-axis with the sign of the ratio determined by the CAST rule.

5. Determine the exact value for each of the following using the special triangles.

a. $\tan 150°$	**b.** $\cos 300°$
c. $\sin(-225°)$	**d.** $\cos(-315°)$
e. $\sin 210°$	**f.** $\tan(-300°)$
g. $\cos(-135°)$	**h.** $\sec 225°$

6. Find the exact value of the following:

a. $(\sin 315°)^2 + (\cos 45°)^2$

b. $2\sec(-45°) - \cot 135°$

c. $\sin 225° \times \cos(-60°)$

d. $2\tan 45° - 6\sec(-240°)$

Check your answers by turning to the Appendix.

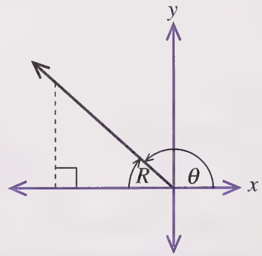
It was necessary to develop the idea of reference angles to simplify more complex angles in trigonometry. Now, with modern technology, such as a scientific calculator, your calculations are done with ease.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

The reference angle is the angle at the origin. It is part of a right triangle which is formed when a perpendicular is drawn from the terminal arm to the x-axis.



There are at least two angles on a coordinate plane which can have the same terminal arm.

Choose the correct response to make each statement true.

1. a. (Coterminal angles, Complementary angles) have the same terminal arm and initial ray.

   b. A negative angle results from a (clockwise, counterclockwise) rotation.

2. For each angle, give the quadrant in which it terminates; then find the related first-quadrant angle.

   a. $\theta = 280°$
   _______ quadrant
   $R = 360° - \theta$
   $= $ _______

   b. $\theta = 140°$
   _______ quadrant
   $R = $ _______

   c. $\theta = 210°$
   _______ quadrant
   $R = $ _______

   d. $\theta = -120°$
   _______ quadrant
   $R = $ _______

   e. $\theta = -190°$
   _______ quadrant
   $R = $ _______

3. In which quadrant does the terminal arm of θ lie for each of the following?

   a. $\sin\theta$ positive and $\cos\theta$ negative
   b. $\tan\theta$ positive and $\cos\theta$ positive
   c. $\cot\theta$ positive and $\sin\theta$ negative
   d. $\sec\theta$ positive and $\sin\theta$ negative
   e. $\csc\theta$ negative and $\cot\theta$ negative

4. Draw each angle; then find the reference angle. Give each ratio correct to four decimal places.

   a. $\sin 110°$ b. $\tan(-140°)$ c. $\sec 325°$

5. Find the angle between 0° and 360° that is coterminal with an angle of $-30°$.

Check your answers by turning to the Appendix.

Enrichment

Some angles can be quite complex.

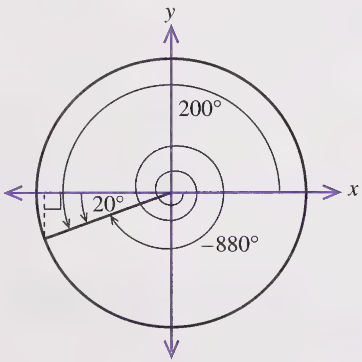
The following example shows you how to find the trigonometric function for a more complex angle.

Example

Using reference angles, find the value of $\tan(-880°)$ to two decimal places.

Solution

Find the positive value of θ that is coterminal with $-880°$. Add $3(360°)$ to $-880°$ to find the angle between $0°$ and $360°$ that is coterminal with $-880°$. That is, $1080° - 880° = 200°$ is coterminal with $-880°$.



$$R = 200° - 180°$$
$$= 20°$$

$$\tan(-880°) = \tan 20°$$

In the third quadrant, tangent is positive.

Using a calculator, you would get the following answers.

2 0 tan

0.363970234

8 8 0 +/− tan

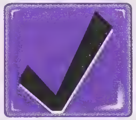
0.363970234

1. Find the exact value of the following:

   a. $\sec(-210°)$ b. $\cot(-150°)$
   c. $\cos 855°$ d. $\csc 600°$

2. Use the reference angle method to find $\sin(-670°)$ correct to three decimal places.

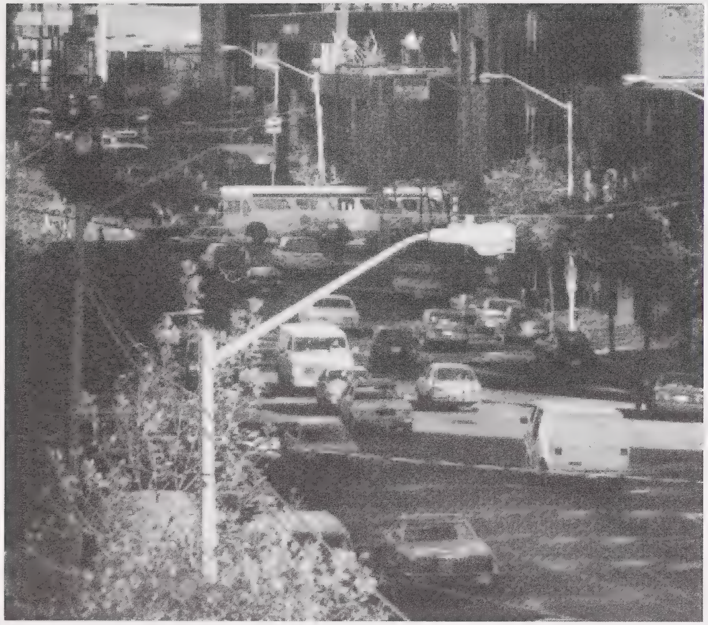
3. Use the reference angle method to find the value of $\cos(-598°)$ correct to four decimal places.



Check your answers by turning to the Appendix.

Conclusion

In this section you defined and evaluated the three primary trigonometric ratios (sine, cosine, and tangent) and their reciprocals (cosecant, secant, and cotangent) for both positive and negative angles drawn in standard position. Using these definitions, you worked backwards from functional values to find the corresponding angles in the plane that yielded those functional values. Also, given the value of one of the six trigonometric functions, you were able to determine any one of the other five trigonometric functions.



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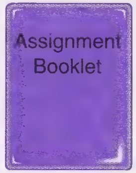
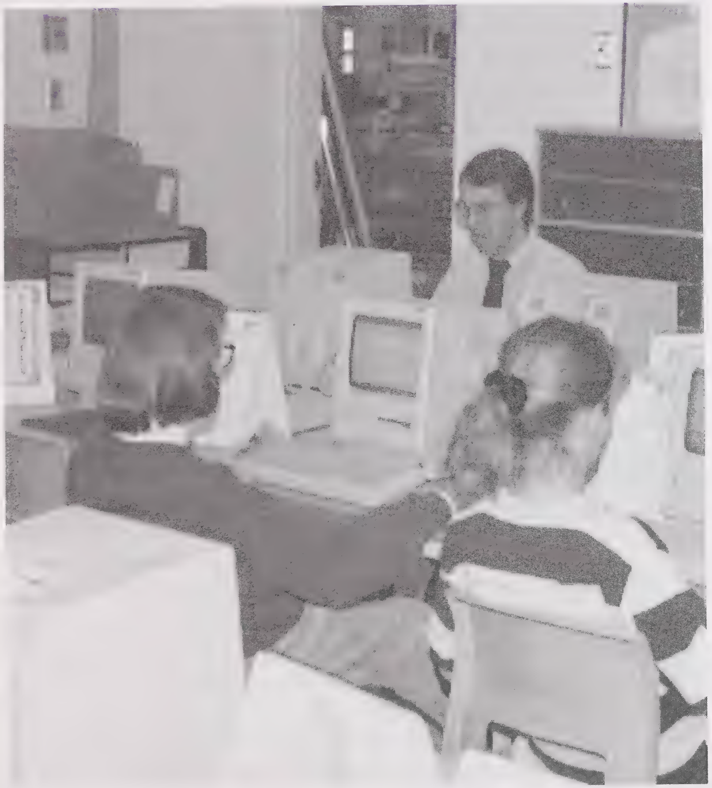
If in the future you choose to continue studying mathematics or any other mathematics-related program, you will discover that trigonometric functions are fundamental to the modern world. Just as in the development of the railroad in the past, the design and construction of transportation arteries today, which are very necessary for nations to function, depends on mathematics and the practical application of trigonometry. Without proper design, the transporting of goods, services, and people could not be done efficiently.

Assignment



You are now ready to complete the section assignment.

Section 2: The Unit Circle



How do calculators and computers simplify your work? When is it appropriate to use mental computation or paper-and-pencil techniques?

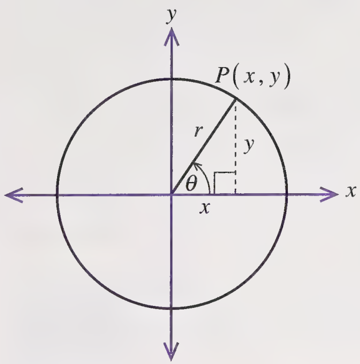
Living in a technological society, calculators and computers dominate the way people work, think, and interact with other people. Yet, even today, it is still important to be able to perform certain functions without having to rely on an external aid.

In this section you will continue your investigation of trigonometric functions. You will also be introduced to the unit circle. Later on in this section, the unit circle will be used to develop exact trigonometric ratios of several special angles measured in either degrees or radians. The unit circle will make it a valuable model to assist you in understanding and learning trigonometric functions. The unit circle will reduce your reliance on the calculator and the computer.

You will also develop an alternate method for measuring angles called **radians**. (Radian measure is particularly suitable for solving problems involving rotational speed.)

Activity 1: Defining Trigonometric Functions Using the Unit Circle

You will recall from the previous section that definitions of the six trigonometric ratios use *x*, *y*, and *r* to represent the sides of a right triangle drawn in a circle.



$x^2 + y^2 = r^2$, where $r > 0$

$\sin\theta = \frac{y}{r}$ $\qquad \csc\theta = \frac{r}{y}$

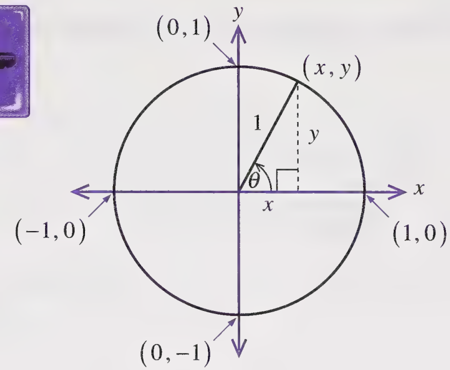
$\cos\theta = \frac{x}{r}$ $\qquad \sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\qquad \cot\theta = \frac{x}{y}$

The hypotenuse of a right triangle is *r* and is always positive.

These definitions may be simplified if a radius of one unit is used. All the points $P(x, y)$ on the terminal arm of the angle lie on a circle with a radius of one unit. This circle is called the **unit circle**.

Now, look at the following diagram. Observe what happens to the trigonometric ratios for the unit circle. For the unit circle, the radius is one unit.



$\sin\theta = \frac{y}{r} = \frac{y}{1} = y$ $\qquad \csc\theta = \frac{1}{y}$

$\cos\theta = \frac{x}{r} = \frac{x}{1} = x$ $\qquad \sec\theta = \frac{1}{x}$

$\tan\theta = \frac{y}{x}$ $\qquad \cot\theta = \frac{x}{y}$

The equation of the unit circle is $x^2 + y^2 = 1$. It describes the relationship between the *x*-coordinate and the *y*-coordinate for **all** points on the unit circle.

Therefore, if $P(x, y)$ is a point on the terminal arm of angle θ and also a point on the unit circle, then the x-coordinate is the value of $\cos\ \theta$ and the y-coordinate is the value of $\sin\ \theta$.

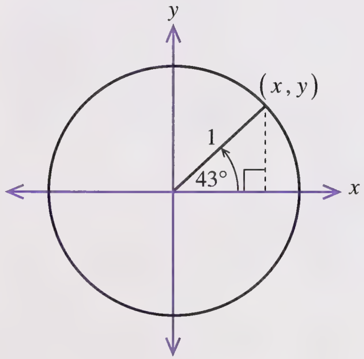
$$\tan\ \theta = \frac{y}{x}$$

The following examples will demonstrate how the points on the unit circle are related to the trigonometric ratios.

Example 1

Determine the coordinates, correct to two decimal places, for a point on the unit circle which is on the terminal arm of 43°.

Solution



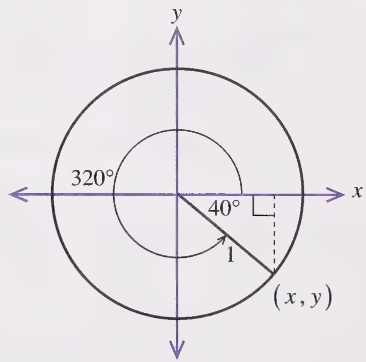
$$(x, y) = (\cos\ 43°, \sin\ 43°)$$
$$\doteq (0.73, 0.68)$$

A scientific calculator may be used to find the coordinates of a point directly. The next example shows this.

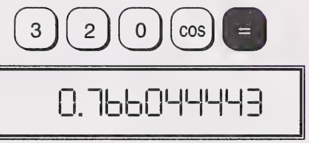
Example 2

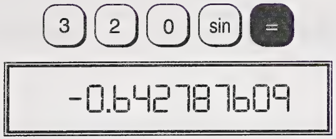
Determine the coordinates, correct to two decimal places, for a point on the unit circle which is on the terminal arm of 320°.

Solution



$$(x, y) = (\cos\ 320°, \sin\ 320°)$$



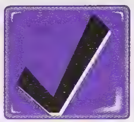


$(x, y) \doteq (0.77, -0.64)$

In the fourth quadrant, the *x*-coordinate is positive and the *y*-coordinate is negative.

1. Find the coordinates of the point on the unit circle, which is also on the terminal arm of the following angles. Evaluate to the nearest tenth.

a. 32° **b.** 287° **c.** −193°



Check your answers by turning to the Appendix.

Example 3 is similar to the questions you did in Section 1. Use the unit circle to obtain the solutions for this example.

Example 3

If $\sin\theta = \frac{3}{5}$ and $90° < \theta < 180°$, determine the values of the other trigonometric functions.

Solution

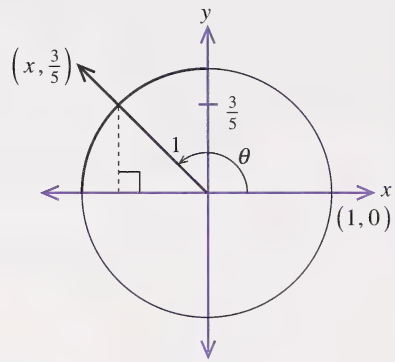
Method 1

Draw a circle with a radius 1 unit. Since $\sin\theta = \frac{3}{5}$, the *y*-coordinate of the corresponding point on the unit circle is $\frac{3}{5}$.

$\left(x, \frac{3}{5}\right)$ is a point on the terminal arm of angle θ. You have to find x using the equation $x^2 + y^2 = 1$.

By substitution,

$$
\begin{aligned}
x^2 + \left(\frac{3}{5}\right)^2 &= 1 \\
x^2 + \frac{9}{25} &= 1 \\
x^2 &= 1 - \frac{9}{25} \\
x^2 &= \frac{25}{25} - \frac{9}{25} \\
x^2 &= \frac{16}{25} \\
x &= \pm\sqrt{\frac{16}{25}} \\
&= \pm\frac{4}{5}
\end{aligned}
$$



$90° < \theta < 180°$ means the terminal arm of θ is in the second quadrant and x must be negative. Thus, $x = -\frac{4}{5}$.

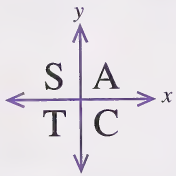
$\therefore \ \sin\theta = \frac{3}{5}$ $\qquad \csc\theta = \frac{5}{3}$

$\cos\theta = -\frac{4}{5}$ $\qquad \sec\theta = -\frac{5}{4}$

$$\tan\theta = \frac{y}{x} = \frac{\frac{3}{5}}{-\frac{4}{5}} = -\frac{3}{4}$$

$\cot\theta = -\frac{4}{3}$

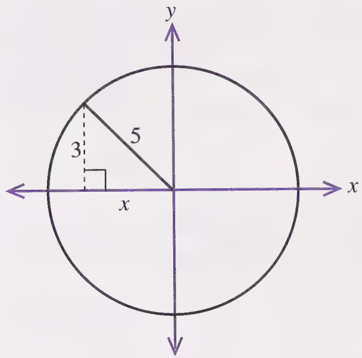
The CAST rule confirms the sign associated with each trigonometric ratio. In the second quadrant, only the sine ratio and its reciprocal ratio (cosecant) will be positive.



Can you recall the alternate method for doing this example from Section 1? Refresh your memory!

Method 2

Since $\sin\theta = \frac{3}{5}$ and $\sin\theta = \frac{y}{r}$, then $y = 3$, $r = 5$, and θ terminates in the second quadrant.



$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + 3^2 &= 5^2 \\ x^2 &= 25 - 9 \\ x^2 &= 16 \\ x &= \pm\sqrt{16} \\ &= \pm 4 \end{aligned}$$

Since θ terminates in the second quadrant, then $x = -4$.

$\therefore \ \cos\theta = \frac{-4}{5}$ $\qquad \tan\theta = \frac{-3}{4}$ $\qquad \sec\theta = \frac{-5}{4}$

$\csc\theta = \frac{5}{3}$ $\qquad \cot\theta = \frac{-4}{3}$

Now you have two methods to use when solving a problem like Example 3. Use the method you prefer.

2. For each of the following, determine the values of all the trigonometric functions given the conditions stated.

a. $\sin\theta = -\frac{5}{13}$, where $270° < \theta < 360°$

b. $\cos\theta = \frac{2}{3}$, where $0° < \theta < 90°$

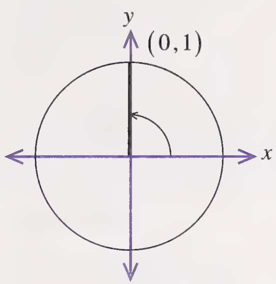
c. $\sec\theta = -2$, where $90° < \theta < 180°$

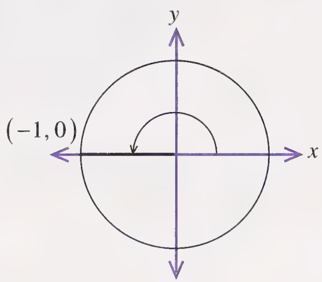
d. $\csc\theta = -\frac{4}{3}$, where $180° < \theta < 270°$

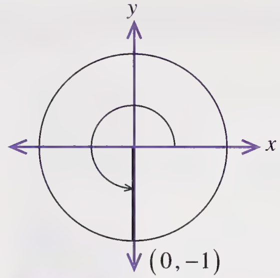
Check your answers by turning to the Appendix.

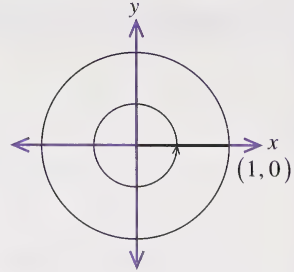
When the angle θ is a multiple of 90°, the terminal arm is on either the x-axis or the y-axis. An angle in which the terminal arm coincides with an axis is called a **quadrantal angle**.

Study the following diagrams which illustrate quadrantal angles.

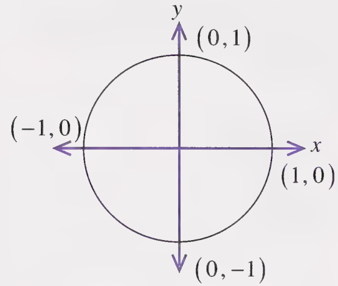








The trigonometric ratios for quadrantal angles can be conveniently determined by remembering the axis intercepts of the unit circle.



Work through the next example which demonstrates this.

Example 4

Evaluate the six trigonometric function values for $\theta = 90°$.

Solution

The point on the unit circle which is also on the terminal arm of a 90° angle in standard position is $(0, 1)$.

$$\therefore \sin 90° = y \\ = 1$$

$$\csc 90° = \frac{1}{y} \\ = \frac{1}{1} \text{ or } 1$$

$$\cos 90° = x \\ = 0$$

$$\sec 90° = \frac{1}{x} \\ = \frac{1}{0} \text{ or undefined}$$

$$\tan 90° = \frac{y}{x} \\ = \frac{1}{0} \text{ or undefined}$$

$$\cot 90° = \frac{x}{y} \\ = \frac{0}{1} \text{ or } 0$$

3. From Example 4, you know the trigonometric function values of 90°. Prepare and complete a chart like the following. Shade in the areas that are undefined.

Quadrantal Angle θ	0°	90°	180°	270°	360°
Point on Terminal Arm		$(0, 1)$			
$\cos\theta$		0			
$\sin\theta$		1			
$\tan\theta$					
$\sec\theta$					
$\csc\theta$		1			
$\cot\theta$		0			

4. Evaluate the following:

a. $\sin(-90°)$ **b.** $\tan(720°)$ **c.** $\csc(-270°)$

5. For a unit circle, $x = \cos\theta$ and $y = \sin\theta$. Will you have the same representation for a circle with radius 2? Explain.

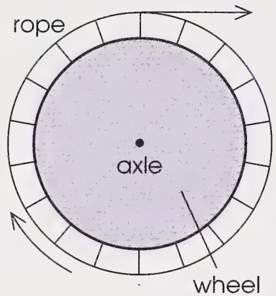
Check your answers by turning to the Appendix.

The CAST rule is also useful in finding trigonometric ratios in a unit circle.

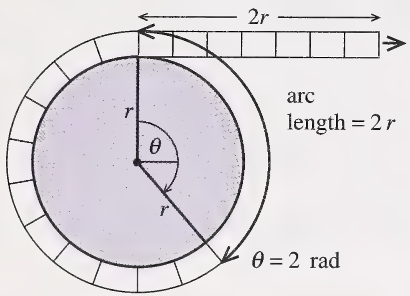
Activity 2: Defining and Using Radian Measure

For many situations involving rotating circular shapes or paths, it is convenient to discuss angles in terms of **radian measure**.

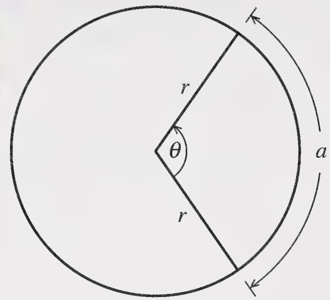
To understand radian measure, consider a wheel rotating around an axle located at the centre. Suppose you wrapped a rope around the rim of the wheel with one end of the rope fixed to the rim. As you pull on the other end of the rope, the wheel rotates freely.



The angle through which the wheel rotates depends upon the length of rope you pull. If the length of rope you pull is equal to the radius of the wheel, then you can say **the wheel rotated through an angle of one radian**. Similarly, if the length of rope you pull is twice as long as the radius, then you can say **the wheel rotates through an angle of two radians**. Therefore, the length of rope you pull, expressed in terms of the radius, determines the angle the wheel rotates in radians. Since the rope was originally wrapped around the rim, a relationship between arc length and angle of rotation occurs.



To illustrate this relationship, consider the following circle with radius r.



$$\text{radian measure of an angle} = \frac{\text{arc length}}{\text{radius}}$$

$$\theta = \frac{a}{r}$$

The arc length is the distance measured along the circle between two points on the circle.

A **radian** can now be formally defined as the measure of the central angle of a circle that subtends or cuts off an arc equal in length to the radius of the circle.

Radian measure is used in higher mathematics, particularly in a branch of mathematics called calculus. If at any time you encounter an angle measure without the units clearly identified, then the angle is assumed to be measured in radians.

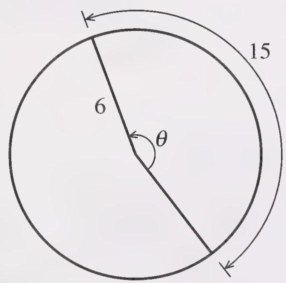
The following example will give you practice in manipulating the formula $\theta = \frac{a}{r}$.

Example 1

Find the value of θ.

Solution

$$\begin{aligned}\theta &= \frac{a}{r}\\ &= \frac{15}{6}\\ &= 2.5\end{aligned}$$



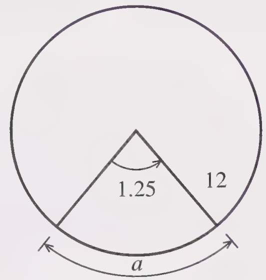
The value of θ is 2.5 rad.

Example 2

Solve for a.

Solution

$$\begin{aligned}\theta &= \frac{a}{r}\\ 1.25 &= \frac{a}{12}\\ 1.25 \times 12 &= a\\ a &= 15 \text{ rad}\end{aligned}$$

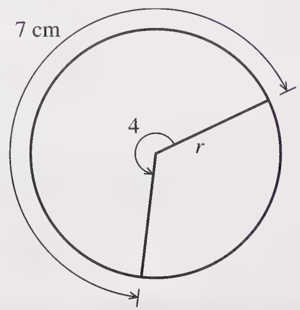


Example 3

If the arc length is 7 cm and the value of θ is 4 radians, find the value of r.

Solution

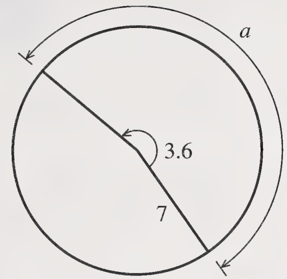
$$\begin{aligned}\theta &= \frac{a}{r}\\ 4 &= \frac{7}{r}\\ r &= \frac{7}{4}\\ &= 1.75\end{aligned}$$



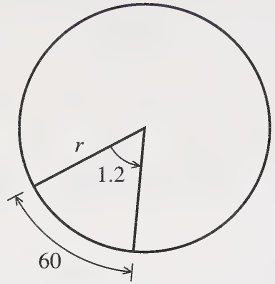
The radius measures 1.75 cm.

1. Find the unknown labelled in each of the following.

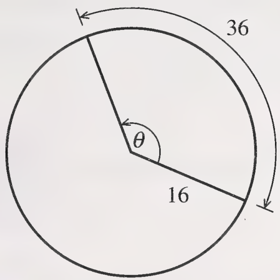
a.



b.



c.



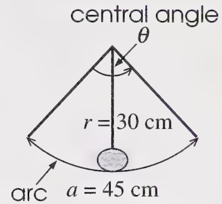
Check your answers by turning to the Appendix.

Look at an application concerning a real-world problem.

Example 4

A pendulum 30 cm long swings through an arc of 45 cm. Through what angle does the pendulum swing?

Solution



The pendulum is 30 cm long. Therefore, $r = 30$ cm.

The arc length a equals 45 cm.

$$\begin{aligned} \therefore\ \theta &= \frac{a}{r} \\ &= \frac{45}{30} \\ &= 1.5 \end{aligned}$$

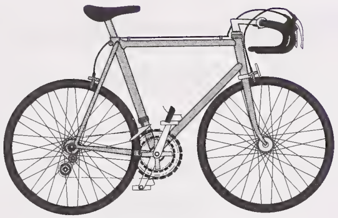
The pendulum swings through an angle of 1.5 rad.

2. The radius of Earth is approximately 6400 km. What is the distance at the equator covered by 2 rad of longitude?

3. A central angle of $\frac{\pi}{2}$ rad in a circle intercepts an arc of the circle. If the length of the intercepted arc is 10 cm, determine the radius of the circle.

4. A satellite circles Earth at a distance of 7500 km from Earth's centre. Give the radian measure if the satellite covered a distance of 12 400 km from a certain location.

5. The spoke of a bicycle wheel is 45 cm in length.

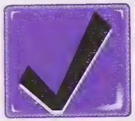


   a. What is the distance travelled by a point on the rim of the wheel if the radian measure is 2.5?

   b. The answer to question 5.a. is an approximation. Why?

6. A pendulum 25 cm long swings through an angle of 1.25 rad.

   a. What is the length of the arc through which the end swings?
   b. Will the angle and arc length change over a period of time?



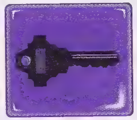
Check your answers by turning to the Appendix.

In this activity you examined radian measure and its application to real-world situations. In Activity 3 you will study the relationship between radian measure and degree measure.

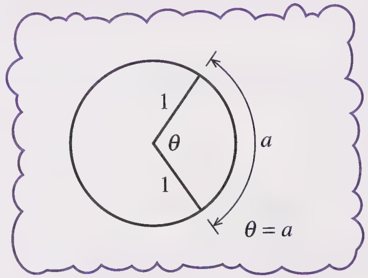
Activity 3: Radian and Degree Measure

The unit circle not only simplifies the definition of the trigonometric functions, it also simplifies a discussion of radian measure.

$$\theta = \frac{a}{r}$$



For a unit circle, $r = 1$; therefore, $\theta = a$.



This means for a unit circle, the radian measure of an angle equals the arc length subtended or cut off by the angle.

You will recall the circumference formula for a circle is $C = 2\pi r$. For a unit circle, $r = 1$; thus, $C = 2\pi(1) = 2\pi$. If you consider an arc length equal to the circumference of the circle, then $\theta = 2\pi$. This establishes the relationship between degrees and radians since an angle of 360° corresponds to the circumference 2π.

$$2\pi \text{ rad} = 360°$$
$$\pi \text{ rad} = 180°$$

$$1 \text{ rad} = \left(\frac{180°}{\pi}\right) \doteq 57.3°$$

$$1° = \left(\frac{\pi}{180}\right) \text{ rad}$$

For another view relating degrees and radians, turn to the Enrichment at the end of Section 2.

You can now convert between radians and degrees as the following examples will illustrate.

Example 1

Convert $\frac{\pi}{5}$ rad to degrees.

Solution

$$1 \text{ rad} = \frac{180°}{\pi}$$

$$\begin{aligned} \therefore \frac{\pi}{5} \text{ rad} &= \frac{180°}{\pi} \times \frac{\pi}{5} \\ &= \frac{180°}{5} \\ &= 36° \end{aligned}$$

You may choose an alternate method.

Since π rad $= 180°$, substitute this into the equation.

$$\begin{aligned} \frac{\pi}{5} \text{ rad} &= \frac{\pi \text{ rad}}{5} \\ &= \frac{180°}{5} \\ &= 36° \end{aligned}$$

Example 2

Convert 3.5 rad to the nearest tenth of a degree. Use a calculator to find the answer. Round your answer to one decimal place.

Solution

Since π rad $= 180°$, then 1 rad $= \frac{180°}{\pi}$.

$$3.5 \text{ rad} = \frac{180°}{\pi} \times 3.5$$

3 • 5 × 1 8 0 ÷ π =

200.5352283

Therefore, $3.5 \text{ rad} \doteq 200.5°$.

Example 3

Convert 120° to an exact radian measure in terms of π.

Solution

Since $180° = \pi$ rad, then $1° = \frac{\pi}{180°}$ rad.

$$120° = 120°\left(\frac{\pi}{180°}\right) \text{ rad}$$
$$= \frac{2\pi}{3} \text{ rad}$$

Reduce; then multiply.

$$\overset{2}{\cancel{120°}}\left(\frac{\pi}{\underset{3}{\cancel{180°}}}\right)$$

Example 4

Convert 146° to the nearest tenth of a radian.

Solution

$$146° = 146°\left(\frac{\pi}{180°}\right) \text{ rad}$$
$$\doteq 2.5481 \text{ rad}$$
$$\doteq 2.5 \text{ rad}$$

Many calculators have the ability to perform radian and degree conversions directly. Refer to the owner's manual for your calculator for details.

Note: $1 \text{ rad} = 1 \times \left(\frac{180°}{\pi}\right)$
$$\doteq 57.296°$$

1. Convert the following degree measures to radian measures. Express your answers as exact values in terms of π, and approximate answers correct to two decimal places.

   a. 45° **b.** $-75°$ **c.** 288° **d.** 1°

2. Convert the following radian measures to degree measures, correct to one decimal place.

   a. $\frac{\pi}{3}$ rad **b.** $-\frac{5\pi}{12}$ rad

   c. -2.7 rad **d.** 5 rad

Check your answers by turning to the Appendix.

Examples 5 and 6 are applications using radian measure.

Example 5

The rim of a wagon wheel with a radius of 60 cm turns through 0.6 rotation. Calculate the angle through which the rim of the wheel turns in radians and in degrees.

Solution

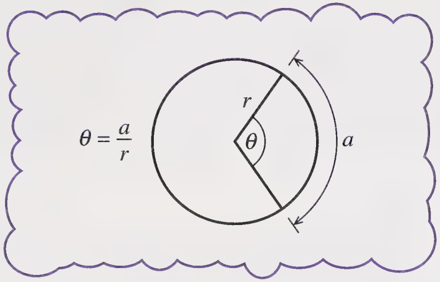
One revolution equals the circumference of the wheel. Find the circumference of the wheel, where $C = 2\pi r$.

$$C = 2\pi(60 \text{ cm})$$
$$= 120\pi \text{ cm}$$

$$0.6 \text{ rotation} = 0.6 \times 120\pi \text{ cm}$$
$$= 72\pi \text{ cm}$$

The arc length for the rim of the wheel is 72π cm.

$$\theta = \frac{a}{r}$$
$$= \frac{72\pi}{60}$$
$$= 1.2\pi$$



$$1 \text{ rad} = \frac{180°}{\pi}$$

$$\therefore\ 1.2\pi \text{ rad} = \frac{180°}{\pi} \times 1.2\pi$$
$$= 216°$$

The rim of the wheel turns 1.2π rad or $216°$.

Example 6

For the wagon wheel described in Example 5, how far does a marker travel if it is located midway between the rim and the centre of the wheel?

Solution

A point halfway between the rim and centre will be located at $r = \frac{60 \text{ cm}}{2} = 30$ cm.

$$\theta = \frac{a}{r}$$
$$1.2\pi = \frac{a}{30 \text{ cm}}$$
$$a = 1.2\pi \times 30 \text{ cm}$$
$$= 36\pi \text{ cm}$$
$$\doteq 113.1 \text{ cm}$$

The marker travels approximately 113.1 cm.

You can study another type of application, called angular velocity, in the Enrichment at the end of Section 2.

3. A circular disk of radius 1 cm is rolled in a straight line along the top of a table. If the disk completes $2\frac{1}{4}$ revolutions, express the distance travelled in terms of π.

4. The arc of a circle with a radius 5 cm measures 16 cm. Find the measure of the central angle in radians and in degrees. Give the degree measure correct to four places.

5. The time on a clock reads 12:20. Calculate the central angle between the two hands in degrees and in radians.

Check your answers by turning to the Appendix.

Now that you have been shown how to convert radian measure to degree measure and vice versa, you will now find the coordinates of the point on a unit circle in either form.

Example 7

Find the coordinates of the point on the unit circle that is also on the terminal arm of an angle in standard position with measure of $\frac{3\pi}{5}$ rad. Express each coordinate of the point correct to two decimal places.

Solution

Method 1

The point has coordinates $\left(\cos \frac{3\pi}{5}, \sin \frac{3\pi}{5}\right)$.

Convert $\frac{3\pi}{5}$ rad to degree measure.

Since $\pi \text{ rad} = 180°$, this type of expression can be easily converted by substitution.

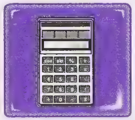
$$\begin{aligned}\frac{3\pi}{5} \text{ rad} &= \frac{3(180°)}{5} \\ &= 108°\end{aligned}$$

The point has coordinates $(\cos 108°, \sin 108°)$. Use a calculator to find the corresponding numerical values to two decimal places.

$$(\cos 108°, \sin 108°) \doteq (-0.31, 0.95)$$

The terminal arm for 108° is in the second quadrant, where the x-coordinate is negative and the y-coordinate is positive.

Method 2



You can determine the values of $\cos \frac{3\pi}{5}$ and $\sin \frac{3\pi}{5}$ more directly using a calculator in radian mode.

Step 1: Change the calculator input mode to RAD. Consult the owner's manual to change the input mode. (Usually RAD or Radian will appear somewhere on the display.)

Step 2: Approximate the value of $\frac{3\pi}{5}$ using the following calculator sequence.

3 × π ÷ 5 =

1.884955592

Step 3: Store this value in the calculator memory.

Step 4: Press cos.

-0.309016994

Step 5: Recall the stored value and press sin.

0.95105651 6

The point has coordinates $(-0.31, 0.95)$.

Note

- Always check your calculator display to determine what input mode (DEG, RAD, GRA) your calculator is in.

- $(-0.31, 0.95)$ is an approximation of the coordinates of the point. This approximation can be improved by including more digits of the calculator display, but the exact value of $\cos\theta$ or $\sin\theta$ cannot be determined using calculators. However, there are some special cases for which the **exact** trigonometric ratios can be determined.

6. Determine the coordinates of the point on the unit circle which is on the terminal arm of the following angles. Evaluate to the nearest hundredth.

   a. $\frac{4\pi}{3}$ rad **b.** $\frac{7\pi}{12}$ rad

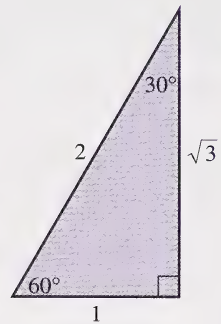
   c. $-\frac{5\pi}{7}$ rad **d.** 1 rad

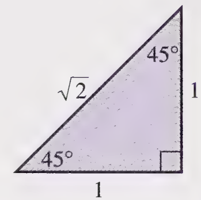
Check your answers by turning to the Appendix.

Your calculator should be very helpful in converting degrees to radian measure and vice versa. Make sure it is in the correct mode when you are working with degrees or radians.

Activity 4: Using Special Angles

You already know the exact value for the ordered pairs associated with the quadrantal angles. Also, from previous work, you know the exact values of the trigonometric ratios for the special angles 30°, 60°, and 45°.





The coordinates of the points on the unit circle, which are also on the terminal arm of these special angles, can be determined exactly.

Study the following example.

Example 1

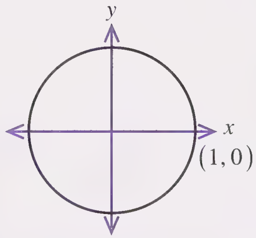
Determine the exact values of the coordinates of the points on the unit circle which are also on the terminal arm of angles with the measures of 0 rad, $\frac{\pi}{6}$ rad, $\frac{\pi}{4}$ rad, $\frac{\pi}{3}$ rad, and $\frac{\pi}{2}$ rad.

Solution

You should know the relationship between these special angles given either degree or radian measure.

$$0 \text{ rad} = 0°$$

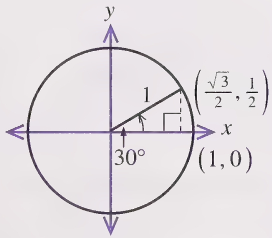
The point on the terminal arm and on the unit circle is $(1, 0)$.

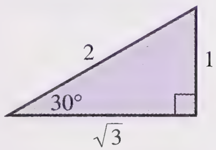


The radius of the circle is 1.

$$\frac{\pi}{6} \text{ rad} = \frac{180°}{6} = 30°$$

The point on the unit circle and the terminal arm of this angle is $(\cos 30°, \sin 30°) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.



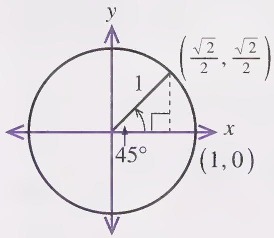


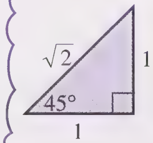
$$\cos 30° = \frac{\sqrt{3}}{2}$$

$$\sin 30° = \frac{1}{2}$$

$$\frac{\pi}{4} \text{ rad} = \frac{180°}{4} = 45°$$

The point on the unit circle and the terminal arm of this angle is $(\cos 45°, \sin 45°) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.



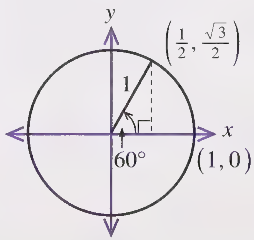


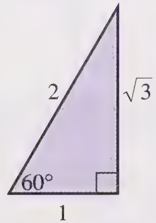
$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\frac{\pi}{3} \text{ rad} = \frac{180°}{3} = 60°$$

The point on the unit circle and the terminal arm of this angle is $(\cos 60°, \sin 60°) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



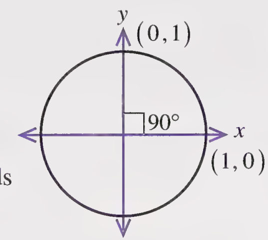


$$\cos 60° = \frac{1}{2}$$

$$\sin 60° = \frac{\sqrt{3}}{2}$$

Note: In each of the triangles in the preceding circles, the side opposite the angle represents the y-coordinate and the adjacent side represents the x-coordinate. For example, in the 60° triangle, $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$.

$$\frac{\pi}{2} \text{ rad} = \frac{180°}{2} = 90°$$



The point on the unit circle that corresponds to this angle has coordinates $(0, 1)$.

1. Complete the following chart.

Angle Measure (degrees)	Angle Measure (radian)	Coordinates
0°	0	$(1, 0)$
30°		$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
	$\frac{\pi}{4}$	
		$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
	$\frac{\pi}{2}$	

Check your answers by turning to the Appendix.

For convenience, it is advantageous to memorize the relationship between radian and degree measures listed in the chart.

To help remember these values, one observation that you might make at this point is that all the ordered pairs corresponding to these special angles will be formed using only numbers from the set $\left\{0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1\right\}$.

Using the coordinates of these angles in the first quadrant, you can determine the coordinates of angles in the other quadrants.

The following examples show this.

Example 2

Find the exact coordinates of the point on the unit circle that is also on the terminal arm of $\frac{5\pi}{6}$ in standard position.

Solution

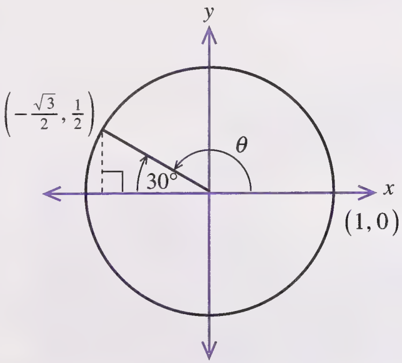
$$\theta = \frac{5\pi}{6}$$
$$= 150°$$

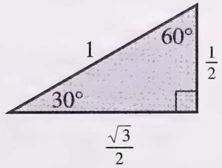
(Another way to perform this conversion is to remember $\frac{\pi}{6} = 30°$; thus, $\frac{5\pi}{6} = 5(30°) = 150°$.)

Follow these steps:

Step 1: Draw a unit circle and the angle accurately in standard position.

Step 2: Using the reference angle, identify the coordinates of the point of intersection of the terminal arm of the angle and the unit circle.





The reference angle is 30°.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

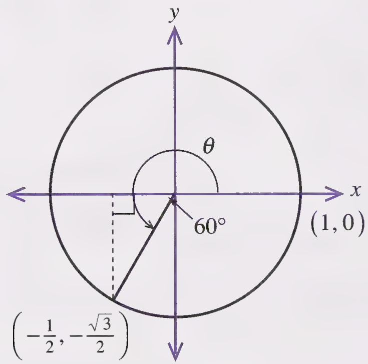
Do you see a relationship between the coordinates for $\frac{\pi}{6}$ in the first quadrant and for $\frac{5\pi}{6}$ in the second quadrant? The sine values are the same but the cosine value is negative in the second quadrant.

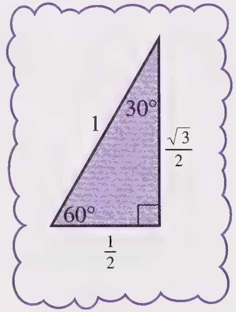
Example 3

Find the exact coordinates of the point on the unit circle that is also on the terminal arm of $\frac{4\pi}{3}$ in standard position.

Solution

$$\theta = \frac{4\pi}{3} \qquad \left(\text{Since } \frac{\pi}{3} = 60°, \text{ then } \frac{4\pi}{3} = 4(60°) = 240°.\right)$$
$$= 240°$$





The reference angle is 60°.

The coordinates of the point are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

In the third quadrant, both coordinates of a point are negative.

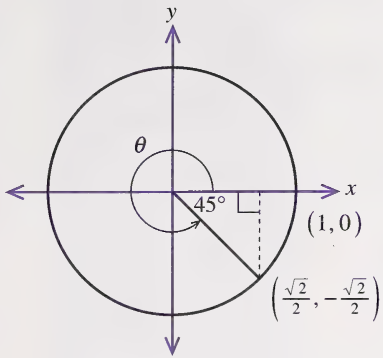
Example 4

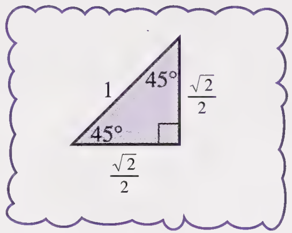
Find the exact coordinates of the point on the unit circle that is also on the terminal arm of $\frac{7\pi}{4}$ in standard position.

Solution

$$\theta = \frac{7\pi}{4} = 315^\circ$$

$\left(\text{Since } \frac{\pi}{4} = 45^\circ, \text{ then } \frac{7\pi}{4} = 7(45^\circ) = 315^\circ.\right)$





The reference angle is 45°.

The coordinates of the point are $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Example 5

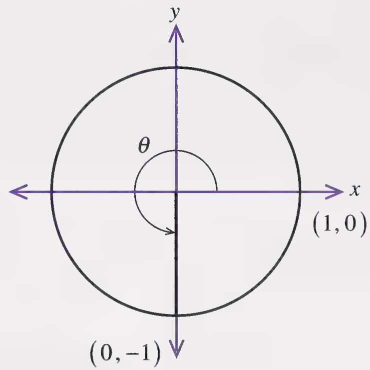
Find the exact coordinates of the point on the unit circle that is also on the terminal arm of $\frac{3\pi}{2}$ in standard position.

Solution

$$\theta = \frac{3\pi}{2} = 270^\circ$$

(Since $\frac{\pi}{2} = 90^\circ$, then $\frac{3\pi}{2} = 3(90^\circ) = 270^\circ$. This is a quadrantal angle.)

The coordinates of the point are $(0, -1)$.

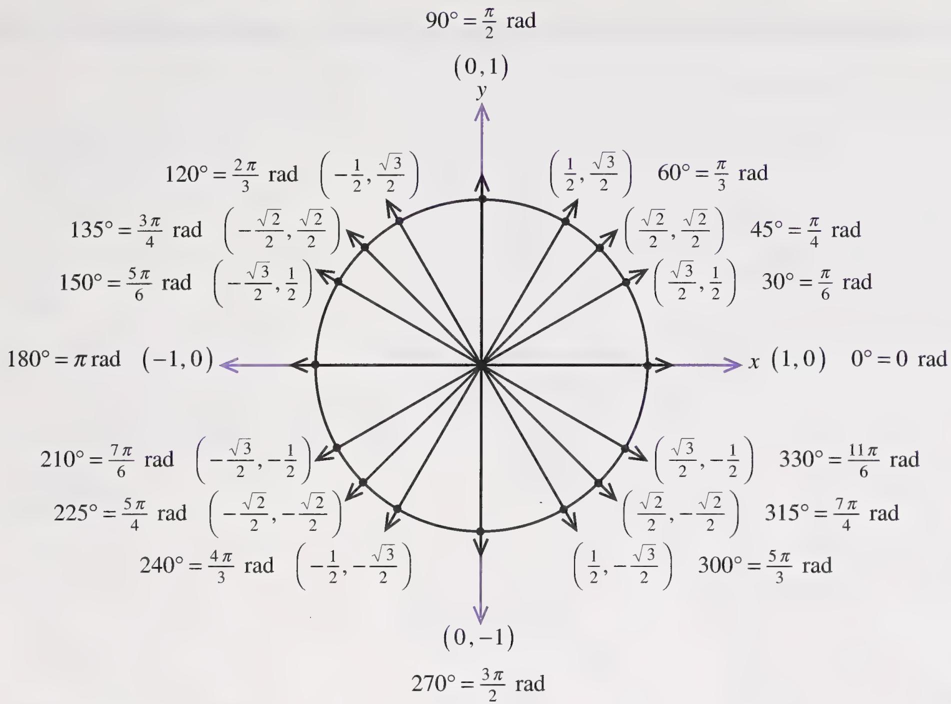


2. Complete a chart like the following to summarize the coordinates of all the special angles with terminal arms in the other three quadrants.

Angle Measure (rad)		Angle Measure (degrees)	Coordinates
	0	0	$(1, 0)$
First Quadrant	$\frac{\pi}{6}$	30	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
	$\frac{\pi}{4}$	45	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
	$\frac{\pi}{3}$	60	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
	$\frac{\pi}{2}$	90	$(0, 1)$
Second Quadrant	$\frac{2\pi}{3}$		
	$\frac{3\pi}{4}$		
	$\frac{5\pi}{6}$		
	π		
Third Quadrant	$\frac{7\pi}{6}$		
	$\frac{5\pi}{4}$		
	$\frac{4\pi}{3}$		
	$\frac{3\pi}{2}$		
Fourth Quadrant	$\frac{5\pi}{3}$		
	$\frac{7\pi}{4}$		
	$\frac{11\pi}{6}$		
	2π		

Check your answers by turning to the Appendix.

The results you obtained in question 2 of this activity can be summarized in the following diagram of the unit circle. One approach to answering questions involving exact values is to memorize this diagram. An easy way is to memorize the first quadrant values and relate these to the other quadrants.



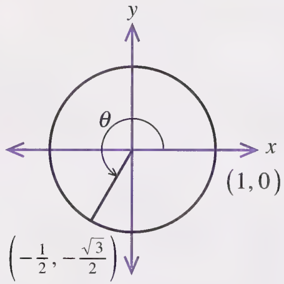
You can now use the coordinates of these special points on the unit circle to determine the exact values of trigonometric ratios of special angles. You can use the radian measure or the degree measure of an angle to find the coordinates of a point.

Example 6

Determine the exact value of $\cos \frac{4\pi}{3}$.

Solution

$$\theta = \frac{4\pi}{3}$$
$$= 240^\circ$$



The reference angle is 60°.

$$\cos \theta = x$$
$$\cos \frac{4\pi}{3} = -\frac{1}{2}$$

Example 7

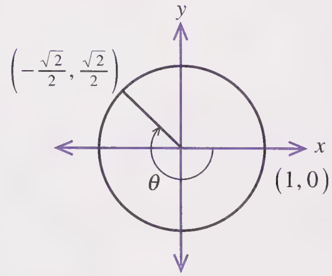
Determine the exact value of $\csc\left(-\frac{5\pi}{4}\right)$.

Solution

$$\theta = -\frac{5\pi}{4}$$
$$= -225^\circ$$

The reference angle is 45°.

$$\csc \theta = \frac{1}{\sin \theta}$$
$$\csc\left(-\frac{5\pi}{4}\right) = \frac{1}{\frac{\sqrt{2}}{2}}$$
$$= 1 \times \frac{2}{\sqrt{2}}$$
$$= \frac{2\sqrt{2}}{2}$$
$$= \sqrt{2}$$



Remember: The values of the trigonometric ratios on the unit circle are as follows:

- $\sin \theta = y$
- $\cos \theta = x$
- $\tan \theta = \frac{y}{x}$
- $\csc \theta = \frac{1}{\sin \theta} = \frac{1}{y}$
- $\sec \theta = \frac{1}{\cos \theta} = \frac{1}{x}$
- $\cot \theta = \frac{1}{\tan \theta} = \frac{x}{y}$

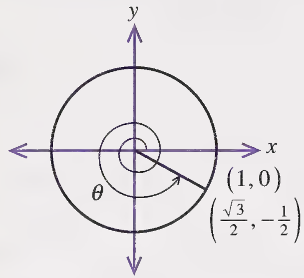
Note: A negative sign can be in any one of the three positions in a fraction: $\frac{-5\pi}{4}$, $\frac{5\pi}{-4}$, or $-\frac{5\pi}{4}$. To be consistent, the negative sign is kept in front of the fraction.

Example 8

Determine the exact value of $\cot \frac{23\pi}{6}$.

Solution

$$\theta = \frac{23\pi}{6}$$
$$= 690°$$



The reference angle is 30°.

$$\cot \theta = \frac{x}{y}$$

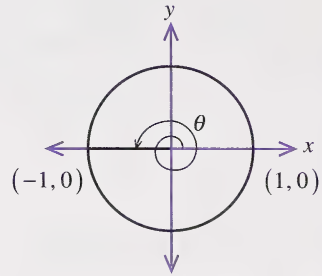
$$\cot \frac{23\pi}{6} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$$
$$= \frac{\sqrt{3}}{2} \times \left(-\frac{2}{1}\right)$$
$$= -\sqrt{3}$$

Example 9

Determine the exact value of $\tan 3\pi$.

Solution

$$\theta = 3\pi$$
$$= 540°$$



$$\tan 3\pi = \frac{y}{x}$$
$$= -\frac{0}{1}$$
$$= 0$$

It is important in learning new information to relate it to concepts which you have learned previously. It is useful to relate the tangent of an angle to the concept of slope from coordinate geometry. The slope of a line is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are the coordinates of two points on the line. Consider the diameter passing through a point on the unit circle. One point on the diameter is the origin $(0, 0)$ and another is the point on the unit circle $P(x, y)$. By substituting (x, y) and $(0, 0)$ into the slope formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, you will get $m = \frac{y-0}{x-0} = \frac{y}{x}$.

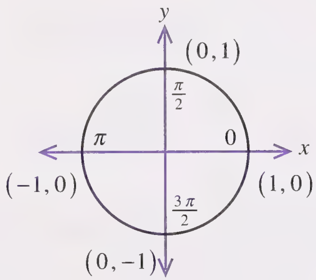
The tangent of an angle is the same as the slope of the diameter of the unit circle which contains the terminal arm of the angle.

Note

The slope of a horizontal line is zero, and the slope of a vertical line is undefined.

$\tan 0 = \tan \pi = 0$

$\tan \frac{\pi}{2}$ and $\tan \frac{3\pi}{2}$ are undefined.



3. Find the exact value of the following:

a. $\sin 120°$ **b.** $\tan(-240°)$

c. $\csc(-300°)$ **d.** $\cos 135°$

4. Find the exact value of the following:

a. $\sin \frac{4\pi}{3}$ **b.** $\sec \frac{7\pi}{6}$ **c.** $\cot\left(-\frac{11\pi}{6}\right)$

d. $\cos \pi$ **e.** $\csc\left(-\frac{7\pi}{4}\right)$

5. Use the unit circle to find the exact value of each.

a. $\sin 225°$ **b.** $\tan \frac{5\pi}{6}$

c. $\csc(-45°)$ **d.** $\tan\left(-\frac{\pi}{3}\right)$

e. $\sec 300°$ **f.** $\sec(-210°)$

g. $\cos\left(-\frac{5\pi}{6}\right)$ **h.** $\cot \frac{5\pi}{4}$

i. $\cos 570°$ **j.** $\tan\left(-\frac{8\pi}{3}\right)$

6. Find the measure of θ, where $0 \le \theta \le 2\pi$.

a. $\tan \theta = -1$ **b.** $\sec \theta = \sqrt{2}$

c. $\sin \theta = -\frac{\sqrt{3}}{2}$ **d.** $\cot \theta = \sqrt{3}$

e. $\cos \theta = \frac{1}{2}$ **f.** $\csc \theta = -\sqrt{2}$

7. Find the value(s) of θ, where $0° \le \theta \le 360°$.

a. $\sin \theta = \frac{\sqrt{3}}{2}$ **b.** $\sec \theta = -\sqrt{2}$

c. $\cos \theta = -1$ **d.** $\cot \theta = 0$

8. Write a general form to show that $\tan \theta$ is undefined when $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

9. Use the unit circle to explain why the values of θ in $\cot\theta = -\sqrt{3}$ are $\frac{5\pi}{6}$ and $\frac{11\pi}{6}$ instead of $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

Use a calculator to answer question 10.

10. Give your answers correct to three decimal places.

a. Find $\sin\left(-\frac{7\pi}{3}\right)$.

b. Find $\tan\left(\frac{11\pi}{15}\right)$.

Check your answers by turning to the Appendix.

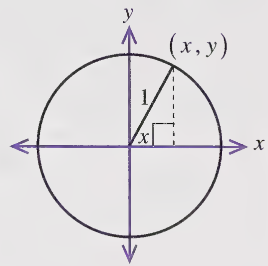
The unit circle uses all the special values of the triangle. Memorize these so that you will not have difficulty using them when exact values are required.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

Here is a summary of the work covered in this section.



- If (x, y) is the terminal point on the unit circle, x represents $\cos\theta$ and y represents $\sin\theta$.

  The radius of a unit circle is 1.

  Therefore, $x^2 + y^2 = 1$.

- Quadrantal angles are angles in which the terminal arm coincides with an axis.

- Radian measure of an angle $= \dfrac{\text{arc length}}{\text{radius}}$. That is, $\theta = \dfrac{a}{r}$.

- $1 \text{ rad} = \frac{180}{\pi}^{\circ}$ and $1^{\circ} = \frac{\pi}{180}$ rad

- A reference angle is an acute angle.

- The unit circle contains special points. The angles are in degrees and radians.

  You can find the coordinates of the points from an angle measure and vice versa.

1. State whether the following statements are True or False.

   a. If $(-2, 5)$ is a terminal point on a circle, then $\sin \theta = 5$.

   b. $x^2 + y^2 = 4$ represents a circle with radius 1.

   c. $450°$ is a quadrantal angle.

   d. The arc length of a circle can be found using the formula $\theta = \frac{a}{r}$, where r is the radius, θ is the angle measure, and a is the arc length.

2. Calculate the unknown value of $\theta = \frac{a}{r}$.

   a. $a = 12$ and $\theta = 125°$ b. $a = 28$ and $r = 6$
   c. $r = 15$ and $\theta = 2.5\pi$

3. Change the following to degrees.

   a. $\frac{\pi}{6}$ rad b. $\frac{7\pi}{12}$ rad
   c. $-\frac{11\pi}{15}$ rad d. $-\frac{8\pi}{3}$ rad

4. Change the following to radians.

   a. $240°$ b. $-135°$
   c. $-540°$ d. $330°$

Use a calculator to answer questions 5 and 6.

5. Find the coordinates of the terminal point on the unit circle for each rotation. Answer correct to three decimal places.

   a. $36°$ b. $215°$ c. $-80°$

6. Change your calculator to radian mode, and find the following values correct to four decimal places.

   a. $\sin \frac{9\pi}{5}$ b. $\cos\left(-\frac{2\pi}{3}\right)$ c. $\tan\left(-\frac{5\pi}{6}\right)$

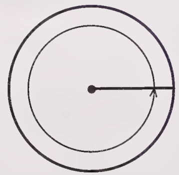
Check your answers by turning to the Appendix.

Enrichment

Here is another way of relating degrees and radians in a circle.

1 revolution $= 360°$

1 revolution = __________ rad



One revolution represents the circumference of the circle.

$C = 2\pi r$

By definition, $\theta = \frac{a}{r}$

$$\begin{aligned} &= \frac{2\pi r}{r} \qquad (a = 2\pi r) \\ &= 2\pi \end{aligned}$$

1 revolution = 360° or 1 revolution = 2π rad.

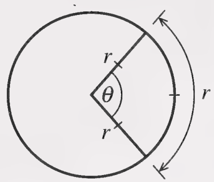
This is true for any circle.

Therefore, 2π rad = 360° and π rad = 180°.

Recall the definition of 1 rad, that is, the central angle formed by an arc equal in length to the radius.

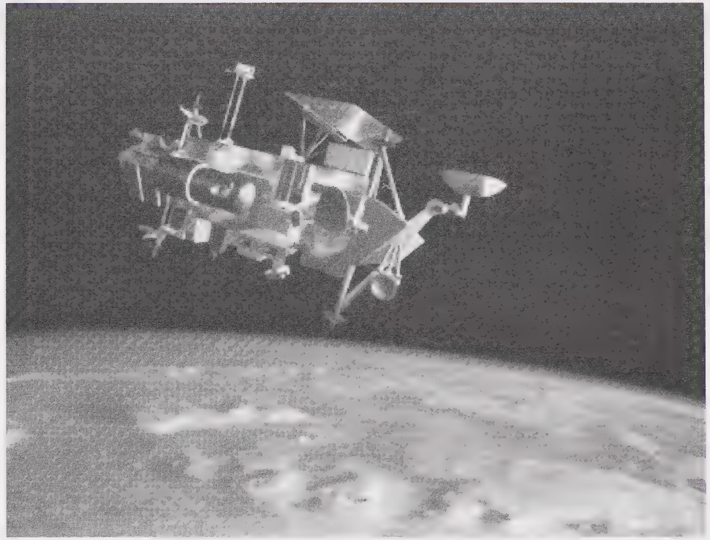
If $a = r$, then $\theta = \frac{a}{r}$

$$\begin{aligned} &= \frac{r}{r} \\ &= 1 \text{ rad} \end{aligned}$$



The sector of the circle formed by 1 rad resembles an equilateral triangle, where each angle = 60°. So, it is logical that θ should measure close to 60°.

While you are dealing with arc length, it is appropriate here to study angular velocity and sector angle.



NASA

A satellite orbits Earth. As time goes on, the number of revolutions increases. When an object rotates around a central point, its velocity is called angular velocity. This velocity is determined by the formula $w = \frac{\theta}{t}$, where w is the angular velocity, θ is the angle of rotation (measured in radians), and t is the time elapsed (in appropriate units).

The angular velocity can be expressed in terms of radians or degrees.

Work through the following examples and see how this formula applies.

Example 1

A wheel revolves at thirty revolutions per minute. What is the angular velocity (in rad/s)?

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Solution

$$1 \text{ rotation} = 360^\circ = 2\pi \text{ rad}$$

$$30 \text{ rotations} = 30 \times 2\pi \text{ rad} = 60\pi \text{ rad}$$

$\theta = 60\pi$ rad and $t = 1$ min $= 60$ s

$$\therefore\ w = \frac{\theta}{t} = \frac{60\pi \text{ rad}}{60 \text{ s}} = \pi \text{ rad/s}$$

The angular velocity is π rad/s.

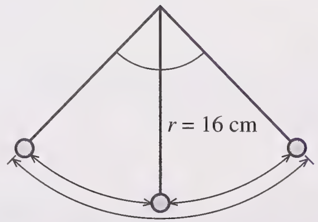
Example 2

A pendulum 16 cm long swings through an arc of 20 cm. If the pendulum takes 0.5 s to swing through the arc, find its average angular velocity.

Solution

First, calculate the radian measure.

$$\theta = \frac{a}{r} = \frac{20}{16} = 1.25 \text{ rad}$$



$$\text{angular velocity} = \frac{\text{amount of rotation}}{\text{time}}$$

$$w = \frac{\theta}{t} = \frac{1.25 \text{ rad}}{0.5 \text{ s}} = 2.5 \text{ rad/s}$$

The average angular velocity is 2.5 rad/s.

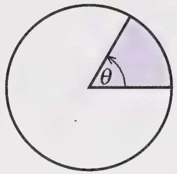
A **sector** of a circle is the region bounded by a central angle of a circle and the arc it intercepts. The area of a sector is dependent on the area of the circle and the measure of the central angle.

Example 3

Determine the sector area of the unit circle in terms of π if the central angle measures $60°$.

Solution

First, find the area of the unit circle.



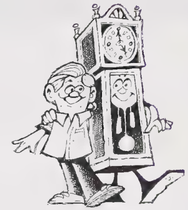
$$\begin{aligned} A &= \pi r^2 \\ &= \pi(1)^2 \\ &= \pi \end{aligned}$$

Now, use the area of the unit circle to determine the sector area.

$$\begin{aligned} \text{sector area} &= \frac{60°}{360°} \times \text{area of unit circle} \\ &= \frac{60°}{360°} \times \pi \\ &= \frac{\pi}{6} \end{aligned}$$

1. A Ferris wheel rotates at a rate of 120 rev/h. Express its angular velocity in rad/min.

2. The angular velocity of a pendulum of a clock is 4.2 rad/s. The length of the pendulum is 30 cm. Calculate the length of the arc through which the end swings if the pendulum takes 0.6 s to swing through the arc.



3. Complete the following chart to determine the sector area in a unit circle determined by a given central angle.

Central Angle (degrees)	Central Angle (radians)	Sector Area
30	$\frac{\pi}{6}$	$\frac{30}{360} \times$ area of unit circle $= \frac{\pi}{12}$
75		
220		
180		

4. Complete the following chart to find the sector area when the measure of the central angle is $100°$ $\left(\text{or } \frac{5\pi}{9} \text{ rad}\right)$.

Radius	Area of the Circle	Area of the Sector
1		
2		
3		
4		

5. Using the results of questions 3 and 4, write a formula for the area of a sector of a circle in terms of the radius of the circle r and the central angle θ (measured in radians).

Check your answers by turning to the Appendix.

Conclusion

In this section you discovered that angles may be measured in either degrees or radians. You examined the relationship between radian measure and degree measure, and determined the values of the trigonometric ratios of angles expressed in either unit. Also, you developed a coordinate system for special angles on the unit circle, angles that are multiples of 30°, 45°, 60°, and 90°. These special angles (and the unit circle) were used to clarify how the values of each trigonometric function are found, and what the relationships among the functions are.

Calculators and computers have removed much of the drudgery from problems in trigonometry. Nevertheless, it is crucial that you can estimate mentally, or use pencil-and-paper techniques, to accept or reject answers obtained using technology. As well, it is imperative that you understand the fundamentals of the discipline. The unit circle provides a mechanism for estimating answers, and for developing and understanding theory.

Both technology and the unit circle assist in problem solving and developing an understanding of the subject of trigonometry.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 3: Solving Equations

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What factors determine how far a baseball player can hit a ball? One of the factors is the speed at which the ball leaves the bat. Another factor, and equally important, is the angle at which the ball rises.

The equation $d = \frac{1}{10} v_0{}^2 \sin 2\theta$ is the formula to determine the distance d the ball will travel, where v_0 is the initial speed at which the ball leaves the bat and θ is the angle at which the ball rises measured from the horizontal. If you know the distance and the speed, you could use this formula to determine the angle at which the ball is hit. This is an example of a trigonometric equation—an equation in which the angle is the unknown.

In this section you will extend your previous knowledge of solving algebraic equations to solving equations involving trigonometric functions. You will solve first-degree and second-degree trigonometric equations involving multiples of angles in the domain $0 \leq \theta \leq 2\pi$. Finally, you will use a calculator to demonstrate the relationships between the root of a trigonometric equation and the graph of the corresponding function.

Activity 1: Solving First-Degree Trigonometric Equations

In solving algebraic equations, the unknown was expressed in terms of a variable and the answer was a real or complex number. In solving trigonometric equations, the unknown is expressed as a trigonometric function, and the answer is expressed as a value of an angle, usually in degree or radian measure.

An example of an algebraic equation is $2x+5=15$.

A similar trigonometric equation might be $2\cos\theta+5=15$.

Notice that the unknown x in the algebraic expression is expressed by a trigonometric function $(\cos\theta)$. However, you are looking for the value of θ, not $\cos\theta$. The equation may contain any of the six trigonometric functions.

Begin by solving a first-degree trigonometric equation.

Example

Solve for θ in $2\sin\theta-1=0$, where $0°\le\theta\le 360°$.

Solution

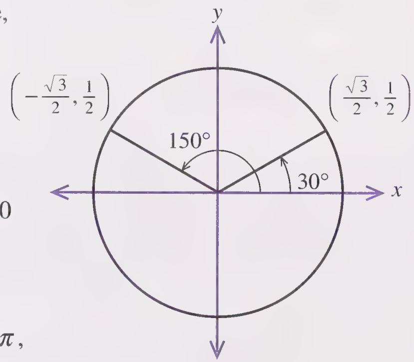
First, notice that there is a restriction on the domain $0°\le\theta\le 360°$ indicating that the value of the angle must be between 0° and 360°.

Isolate the trigonometric function by using your skills of solving algebraic equations.

$$\begin{aligned} 2\sin\theta-1&=0\\ 2\sin\theta&=1\\ \sin\theta&=\frac{1}{2}\end{aligned}$$

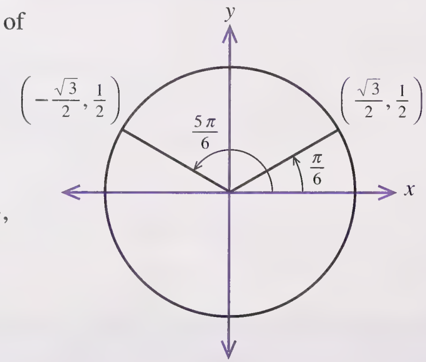
You can now recognize the function in terms of a ratio, and you can determine a possible value of the angle by using the unit circle.

By examining the unit circle, notice that $\sin\theta=\frac{1}{2}$ only occurs at $\theta=30°$ and $\theta=150°$.



Therefore, for $0°\le\theta\le 360°$, $2\sin\theta-1=0$ at $\theta=30°$ and $\theta=150°$.

But, if you were given $2\sin\theta-1=0$ for $0\le\theta\le 2\pi$, then you would be expected to solve the equation in terms of radian measure.



Using the radian measure scale on the unit circle (in Section 2: Activity 4), you find that $\sin\theta=+\frac{1}{2}$, $\theta=\frac{\pi}{6}$, and $\theta=\frac{5\pi}{6}$.

Recall that (x, y) on the unit circle represents $(\cos\ \theta, \sin\ \theta)$.

1. Solve for θ, where $0 \le \theta \le 2\pi$.

a. $\sin\ \theta = -\frac{\sqrt{3}}{2}$ **b.** $\sin\ \theta = \pm\frac{\sqrt{2}}{2}$

c. $\sin\ \theta = 0$

2. Solve for θ, where $0 \le \theta \le \pi$.

a. $\cos\ \theta = -\frac{1}{2}$ **b.** $\cos\ \theta = \pm\frac{\sqrt{2}}{2}$

c. $\cos\ \theta = \pm\frac{\sqrt{3}}{2}$

3. Solve for θ, where $0 \le \theta \le \frac{3\pi}{2}$.

a. $\tan\ \theta = \pm\sqrt{3}$ **b.** $\csc\ \theta = \frac{2}{\sqrt{3}}$

c. $\sec\ \theta = -2$

4. Solve for x, where $0° \le x \le 360°$.

a. $2\cos\ x = -1$ **b.** $5\cot\ x + 5 = 0$

c. $2\sin\ x - \sqrt{3} = 0$

Use a calculator to answer the next question.

5. Solve for $-270° \le x \le -90°$.

a. $2\cos(-x) = -1$ **b.** $\sqrt{3}\csc\ x - 2 = 0$

c. $4\sin(-x) + 1 = 0$ **d.** $-2(\sin\ x + 2) + 3 = 0$

Check your answers by turning to the Appendix.

You must remember that the domain is always given when a solution is required. If the domain is not known, then you have an infinite number of solutions.

Activity 2: Solving Second-Degree Trigonometric Equations

You can also solve second-degree equations of the following types:

- $\sin^2\ \theta = \frac{3}{4}$
- $\sin^2\ \theta - \sin\ \theta = 0$
- $2\cos^2\ \theta - 7\cos\ \theta + 3 = 0$

Now, study the following example.

Example 1

Solve $\sin^2\theta = \frac{1}{2}$, where $0 \le \theta \le 2\pi$.

Solution

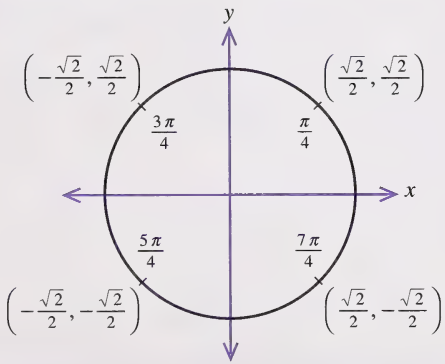
$$\sin^2\theta = \frac{1}{2}$$

$$\sin\theta = \pm\sqrt{\frac{1}{2}} \quad \text{(Take the square root of both sides.)}$$

$$\sin\theta = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

Remember that whenever you take the square root of a number, there are two possible outcomes: $(+)$ and $(-)$.

Now use the unit circle to find the value of θ.



$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ and } \frac{7\pi}{4}$$

Therefore, for $0 \le \theta \le 2\pi$, $\sin^2\theta = \frac{1}{2}$ at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$.

1. Solve for θ, where $0 \le \theta \le 2\pi$.

a. $\sin^2\theta = \frac{3}{4}$ **b.** $2\tan^2\theta - 2 = 0$

c. $\cos^2\theta - \frac{1}{4} = 0$ **d.** $\sec^2\theta = 1$

Check your answers by turning to the Appendix.

Example 2 shows how to solve a second type of trigonometric equation.

Example 2

Solve $\sin^2\theta - \sin\theta = 0$, where $0 \le \theta \le 2\pi$.

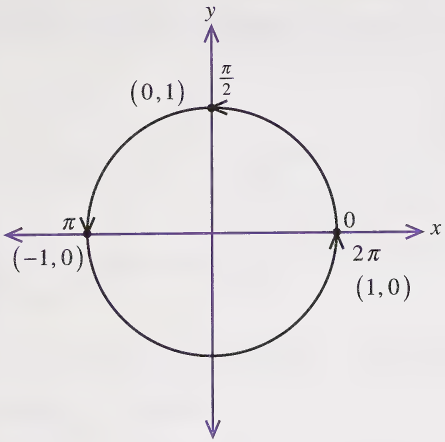
Solution

$$\sin^2\theta - \sin\theta = 0$$

By factoring, $\sin\theta(\sin\theta - 1) = 0$.

$$\sin\theta = 0 \text{ or } \sin\theta - 1 = 0$$
$$\sin\theta = 1$$

Note: In this case, always factor. If you divide by $\sin\theta$, for example, you will lose some possible answers.



Using the unit circle, you can see that $\sin\theta = 0$ when θ is 0, π, or 2π; and $\sin\theta = 1$ when θ is $\frac{\pi}{2}$.

Therefore, for $0 \le \theta \le 2\pi$, $\sin^2\theta - \sin\theta = 0$ at $\theta = 0$, $\frac{\pi}{2}$, π, and 2π.

There are four answers because one trigonometric equation has three possible solutions, and the other trigonometric equation has one solution.

If you want to use another method for factoring trigonometric functions, work through the Enrichment at the end of Section 3.

2. Solve for θ, where $\frac{\pi}{2} \le \theta \le 2\pi$.

a. $\sin\theta \cos\theta - \sin\theta = 0$

b. $\cot^2\theta = \cot\theta$

c. $\tan^2\theta - \tan\theta = 0$

d. $\cos\theta \csc\theta - 2\cos\theta = 0$

Check your answers by turning to the Appendix.

The next example shows you how to solve another type of trigonometric equation. This type has three terms.

Example 3

Solve $2\cos^2\theta - 7\cos\theta + 3 = 0$, where $0 \le \theta \le 2\pi$.

Solution

$$2\cos^2\theta - 7\cos\theta + 3 = 0$$
$$(2\cos\theta - 1)(\cos\theta - 3) = 0$$

$$2\cos\theta - 1 = 0 \quad \text{or} \quad \cos\theta - 3 = 0$$
$$2\cos\theta = 1 \qquad\qquad \cos\theta = 3$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3} \text{ and } \frac{5\pi}{3}$$

Since the maximum value of $\cos\theta$ is 1, the value $\cos\theta = 3$ is not allowed because it does not have a solution.

Therefore, for $0 \le \theta \le 2\pi$, $2\cos^2\theta - 7\cos\theta + 3 = 0$ at $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.

For another method which can be used to factorize this trinomial, go to the Extra Help at the end of Section 3.

For some parts in the following questions, use the fact that $\sin^2\theta + \cos^2\theta = 1$.

3. Solve for x, where $0° \le x \le 360°$.

a. $2\sin^2 x - 3\sin x + 1 = 0$ **b.** $\sin^2 x - \cos^2 x = 0$

4. Solve for θ, where $0 \le \theta \le 2\pi$.

a. $2\sin^2\theta + \sin\theta - 1 = 0$ **b.** $6\cos^2\theta + 5\cos\theta - 4 = 0$

5. Solve for θ, where $0 \le \theta \le 2\pi$.

a. $(1 - \tan\theta)(1 + \cos\theta) = 0$

b. $\sin^2\theta - 2\sin\theta\cos\theta = -\cos^2\theta$

Check your answers by turning to the Appendix.

Look at a case involving real-world situations.

Tides are a periodic rise and fall of the water in the oceans. They are caused by the gravitational attraction of the moon and sun. The difference in the depth of the water between high tide and low tide is so great that boats, such as the one in the photograph, must wait until high tide before entering the water.

The depth of the water h (in metres) can be calculated at any time t (in hours) on the 24-hour clock from the formula

$h = 3\cos 2\pi\frac{(t-4.5)}{12.4} + 5$.

Example 4

Calculate the depth of the water at 7:45 P.M. to the nearest metre.

Solution

Convert the time to decimal hours on the 24-hour clock.

$$\begin{aligned} t &= 7{:}45 \text{ P.M.} \\ &= (12 + 7.75)\text{ h} \\ &= 19.75\text{ h} \end{aligned}$$

Substitute $t = 19.75$ h in the equation.

$$\begin{aligned} h &= 3\cos 2\pi\frac{(t-4.5)}{12.4}+5 \\ &= 3\cos 2\pi\frac{(19.75-4.5)}{12.4}+5 \\ &= 3\cos 2\pi\frac{(15.25)}{12.4}+5 \\ &\doteq 5 \end{aligned}$$

Use a scientific calculator in radian mode and calculate.

At 7:45 P.M., the depth of the water is 5 m.

6. Use the information from Example 4 to do the following:

a. Calculate the depth of the water to the nearest tenth of a metre at 6:30 A.M.

b. Calculate the time of the day when the depth of the water is approximately 7.5 m.

7. The height of a pedal of a bicycle above the ground is given by the equation $h = 15\cos\frac{2\pi t}{5}+24$ where h is height (in centimetres) and t is time (in seconds). Use this equation to determine the height of the pedal, to the nearest tenth of a metre, after 8 s if the topmost position of the pedal starts at $t=0$.

Check your answers by turning to the Appendix.

Most of the time, a second-degree equation has to be factored before solving. Find the exact values using the unit circle, unless you are asked otherwise.

Activity 3: Solving Multiple-Angle Trigonometric Equations

You have seen earlier in your study of trigonometry that it is possible to work with functions of multiple angles, for example, $\sin 2\theta$. Look at an example of this kind.

Example

Solve $2\sin(2\theta)+1=0$, where $0 \le \theta \le 2\pi$.

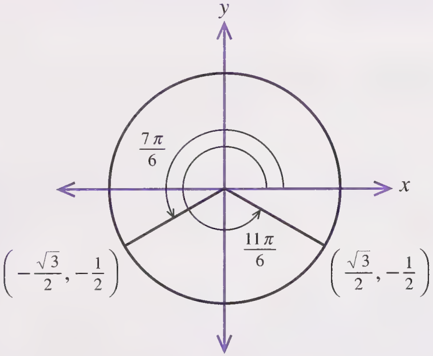
Solution

$$\begin{aligned} 2\sin(2\theta)+1&=0 \\ \sin 2\theta &= -\frac{1}{2} \end{aligned}$$

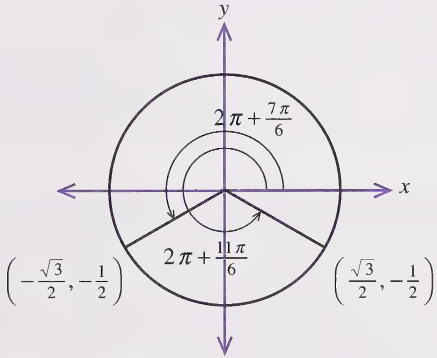
You do not have any method to this point to solve for $\sin 2\theta$. But, if you let $\alpha = 2\theta$, notice that α is twice the size of θ. Therefore, the domain of α will also be twice as large as the domain of θ.

By substituting α for 2θ, then $\sin\alpha = -\frac{1}{2}$ for $0 \le \alpha \le 4\pi$. You can now solve for α, but since $\alpha = 2\theta$, you must consider all values for two complete rotations of the unit circle.

First Rotation



Second Rotation



θ is found within one revolution.
2θ is found within two revolutions.

$$\alpha = \frac{7\pi}{6},\ \frac{11\pi}{6},\ 2\pi + \frac{7\pi}{6},\text{ and } 2\pi + \frac{11\pi}{6}$$
$$= \frac{7\pi}{6},\ \frac{11\pi}{6},\ \frac{19\pi}{6},\text{ and } \frac{23\pi}{6}$$

Since $\alpha = 2\theta$, then $2\theta = \frac{7\pi}{6}$, $\frac{11\pi}{6}$, $\frac{19\pi}{6}$, and $\frac{23\pi}{6}$.

Divide all values of 2θ by 2.

You need the value of θ.

$$\therefore\ \theta = \frac{7\pi}{12},\ \frac{11\pi}{12},\ \frac{19\pi}{12},\text{ and } \frac{23\pi}{12}$$

Therefore, for $0 \le \theta \le 2\pi$, $2\sin(2\theta) + 1 = 0$ at $\theta = \frac{7\pi}{12}$, $\frac{11\pi}{12}$, $\frac{19\pi}{12}$, and $\frac{23\pi}{12}$.

You can check this by substituting $\theta = \frac{7\pi}{12}$ into $2\sin(2\theta) + 1 = 0$.

LS	RS
$2\sin(2\theta)+1$	0
$= 2\sin 2\left(\frac{7\pi}{12}\right)+1$	
$= 2\sin\frac{7\pi}{6}+1$	
$= 2\left(-\frac{1}{2}\right)+1$	
$= -1+1$	
$= 0$	

LS = RS

This check can also be done for $\theta = \frac{11\pi}{12}$, $\frac{19\pi}{12}$, and $\frac{23\pi}{12}$. You should now be able to solve first- and second-degree trigonometric equations involving multiple angles.

1. Use the unit circle to solve the following equations, where $0 \le \theta \le 2\pi$.

   a. $\sin 2\theta = 1$
   b. $\cos 2\theta = 1$
   c. $\tan 2\theta = 1$
   d. $\sqrt{2} \sin 2\theta = 1$

2. Solve for θ when $0° \le \theta \le 360°$.

   a. $4 \sin^2 2\theta = 1$
   b. $2 \sin^2 2\theta - 1 = -\sin 2\theta$
   c. $\tan 3\theta = \sqrt{3}$
   d. $\sin\left(\frac{1}{2}\theta\right) = -\frac{1}{2}$

3. The equation $d = \frac{1}{10} v_0{}^2 \sin 2\theta$ is the formula for the distance d (in metres) a projectile travels and v_0 is the initial speed (in metres per second). If d is 120 m and v_0 is 40 m/s, solve for θ where $0 \le \theta \le 180°$. Give your answer to one decimal place.

Check your answers by turning to the Appendix.

Make sure you understand the calculations involved in finding all the angles in the types of questions in this activity.

Activity 4: Solving Trigonometric Equations Graphically

You know how to solve trigonometric equations by using the idea of solving algebraic equations. In this activity, you will see the relationship between the root of a trigonometric equation and the graph of the corresponding function.

Study the following example.

Example

Solve the equation $\sin \theta - \frac{1}{2} = 0$ graphically, where $0 \le \theta \le 2\pi$.

Solution

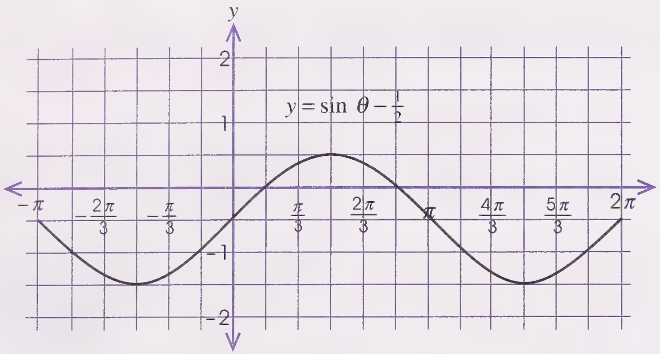
Method 1

Let $y = \sin \theta - \frac{1}{2}$.

Graph this function.

θ	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$
y	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0

θ	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
y	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-1	$-\frac{1}{2}$



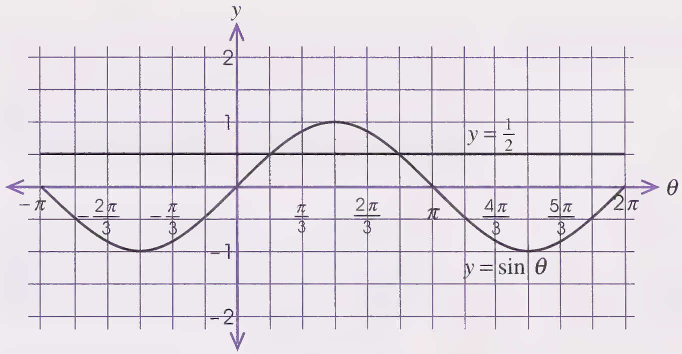
Look at $y = 0$. The x-intercepts in the interval between 0 and 2π are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. These are the solutions in the given domain.

Method 2

Graph $y = \sin\,\theta$ and $y = \frac{1}{2}$.

$\boldsymbol{\theta}$	$-\pi$	$-\frac{5\pi}{6}$	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$
$\boldsymbol{y = \sin\,\theta}$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$

$\boldsymbol{\theta}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$\boldsymbol{y = \sin\,\theta}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	0



Look at the graphs of $y = \frac{1}{2}$ and $y = \sin\,\theta$. The intercepts at $y = \frac{1}{2}$ and $y = \sin\,\theta$, in the interval between 0 and 2π, are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. These are the solutions in the given domain.

You may use the Zap-a-Graph™ to check the accuracy of the preceding graphs.

1. a. Determine the roots of the equation $4\cos^2\theta - 1 = 0$ graphically in the interval $-\pi \le \theta \le \pi$.

   b. Determine the roots of the equation $4\cos^2\theta - 1 = 0$ algebraically in the interval $-\pi \le \theta \le \pi$.

   c. Compare the graphical solutions with the algebraic solutions.

Use a graphing calculator to answer question 2.

2. **a.** Solve $\sin x - \cos x = 0$, where $0 \le x \le 360°$.

b. Give a general form of the solutions.

Check your answers by turning to the Appendix.

You have solved trigonometric equations graphically and algebraically. You should also be familiar with finding the approximate values of the roots using a graphing calculator.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

In this section, you solved the following types of trigonometric equations: first-degree equations, second-degree equations, and multiple-angle equations.

Use the following example to help solve the questions in Extra Help.

Example

Solve $2\cos^2\theta - 7\cos\theta + 3 = 0$ for all values of θ such that $0 \le \theta \le 2\pi$.

Solution

Recall from the work with the unit circle that $x = \cos\theta$.

Substitute $x = \cos\theta$ into $2\cos^2\theta - 7\cos\theta + 3 = 0$.

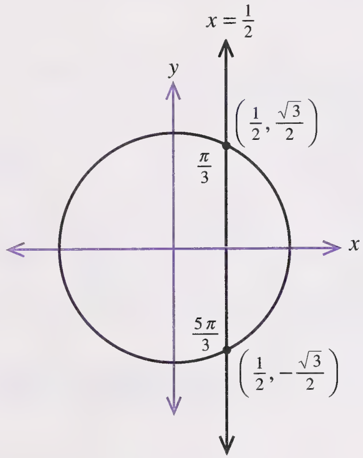
$$2x^2 - 7x + 3 = 0$$
$$(2x-1)(x-3) = 0$$
$$2x - 1 = 0 \quad \text{or} \quad x - 3 = 0$$
$$2x = 1 \qquad\qquad x = 3$$
$$x = \frac{1}{2}$$

When $x = \frac{1}{2}$, then $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$. Refer to the unit circle where $x = \frac{1}{2}$ intersects the unit circle.

Since the maximum value of $\cos\theta$ is 1, there are no values of θ such that $\cos\theta = 3$. Thus, this possible solution is discarded.

Therefore, the only solution is $\cos\theta = \frac{1}{2}$.

From the unit circle, you see that $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.



Solve the following trigonometric equations, where $0 \le \theta \le 2\pi$.

1. $\tan \theta = 1$ (Hint: Express $\tan \theta$ in terms of x and y.)

2. $\cos \theta = -1$

3. $\sin^3 \theta = -1$ (Hint: $\sqrt[3]{-1} = -1$)

4. $\tan^2 \theta = 1$ (Hint: Express $\tan \theta$ in terms of x and y.)

5. $5 + 5\cos \theta = 3\cos \theta + 4$

6. $2\sin^2 \theta + \sin \theta = 0$

7. $\tan^2 \theta - \tan \theta = 0$

8. $2\sin^2 \theta + 7\sin \theta - 4 = 0$ (Hint: Change $\sin \theta$ to y and then factor.)

9. $\sin \theta \times \cos \theta + \sin \theta + \cos \theta + 1 = 0$ (Hint: Use $\cos \theta = x$ and $\sin \theta = y$.)

Check your answers by turning to the Appendix.

Enrichment

There is another method that can be used to solve trigonometric equations.

A trigonometric equation usually has an infinite number of roots. Sometimes, the roots are restricted to a particular domain. The unit circle deals with specific coordinates and θ-values. The following example shows how to solve a trigonometric equation that cannot be factored. The θ-values are not specific to any of the special coordinates in the unit circle.

Example

Solve the following equation for θ to one decimal place.

$$2\sin^2 \theta + \sin \theta - 2 = 0, \text{ where } 0 \le \theta \le 360°$$

Solution

Since $2\sin^2\theta + \sin\theta - 2 = 0$ is a quadratic equation, use the quadratic formula to solve for θ.

$$\sin\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sin\theta = \frac{-1 \pm \sqrt{(1)^2 - 4(2)(-2)}}{2(2)}$$

$$\sin\theta = \frac{-1 \pm \sqrt{17}}{4}$$

$$\sin\theta \doteq 0.780\,776\,4 \quad \text{or} \quad \sin\theta = -1.280\,776$$

$$\theta \doteq 51.3^\circ$$

Another root of $\sin\theta$ is $180^\circ - 51.3^\circ = 128.7^\circ$.

No real values of θ satisfy $\sin\theta = -1.280\,776$. Therefore, $2\sin^2\theta + \sin\theta - 2 = 0$, for the interval $0 \le \theta \le 360^\circ$, at $\theta \doteq 51.3^\circ$ and 128.7°.

Solve each of the following to two decimal places.

1. $-2\cos^2\theta + 2\cos\theta + 3 = 0$, where $0 \le \theta \le 360^\circ$

2. $3\tan^2 x + \tan x - 1 = 0$, where $0 \le x \le 360^\circ$

3. $2\sin^2\theta - 4\sin\theta - 1 = 0$, where $180^\circ \le \theta \le 360^\circ$

Check your answers by turning to the Appendix.

Conclusion

In this section you used your knowledge of algebraic equations to help solve trigonometric equations involving multiples of an unknown angle.

One of the applications of the trigonometric equation involves the trajectory of a projectile. Whether that projectile is an arrow or a baseball, the underlying principles that describe the path it follows are the same. When you and a friend toss a ball back and forth, you instinctively decide what the angle should be when the ball is released, and what speed will be necessary for the ball to arc toward your friend. A mathematician, knowing how far away your friend is and what speed you usually throw the ball, could calculate the angle (to the horizontal) at which you must release the ball using the techniques of this section.

Assignment

Assignment Booklet

You are now ready to complete the section assignment.

Section 4: Graphs of Trigonometric Functions

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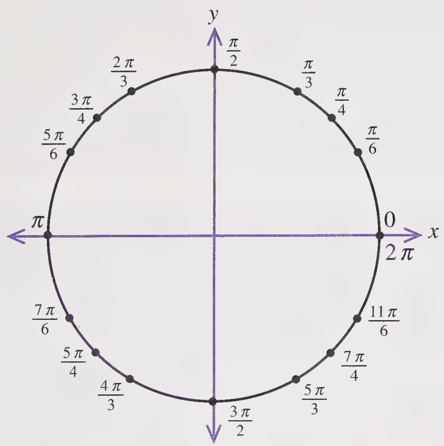
Have you ever watched rolling waves while relaxing on a beach or at the wavepool at West Edmonton Mall? Waves rolling onto a beach are representative of recurring or periodic events. This and other periodic phenomena may be described using trigonometric functions and may be modelled by the graphs of those functions. The transmission of light and sound, the motion of tides, the swinging of a pendulum, and the recurring path of a rock tied to a string and swung in a circle are but a few of the everyday situations that may be understood more clearly using trigonometric functions and their graphs.

In this section you will extend your understanding of trigonometry to include graphic representations of the six basic trigonometric functions. You will investigate transformations of the sine and cosine graphs. Calculators or computers will be used to draw and analyse graphs of trigonometric functions. You will also investigate the effects of the parameters *a*, *b*, *c*, and *d* on the graphs of trigonometric functions using calculators or computers. Finally, you will find the domain and range of all the trigonometric functions.

Activity 1: Graphing Basic Trigonometric Functions

In previous studies of the unit circle, you were shown that the values of the trigonometric functions change as the value of the angles change.



Use the unit circle to examine the value of $y = \sin\ \theta$ for the special angles between $0° \leq \theta \leq 180°$ or $0 \leq \theta \leq \pi$.

θ		$y = \sin\ \theta$	
degrees	radians	exact	approximate
0	0	0	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	0.5
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	0.7
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	0.9
90	$\frac{\pi}{2}$	1	1
120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	0.9
135	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	0.7
150	$\frac{5\pi}{6}$	$\frac{1}{2}$	0.5
180	π	0	0

Now examine the value of $y = \sin\ \theta$ for the special angles between $180° < \theta \leq 360°$ or $\pi < \theta \leq 2\pi$.

θ		$y = \sin\ \theta$	
degrees	radians	exact	approximate
210	$\frac{7\pi}{6}$	$-\frac{1}{2}$	−0.5
225	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	−0.7
240	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	−0.9
270	$\frac{3\pi}{2}$	−1	−1
300	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	−0.9
315	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	−0.7
330	$\frac{11\pi}{6}$	$-\frac{1}{2}$	−0.5
360	2π	0	0

You should begin to get a feeling for the periodic nature of the sine function by observing the recurrence of some of the values for different angles.

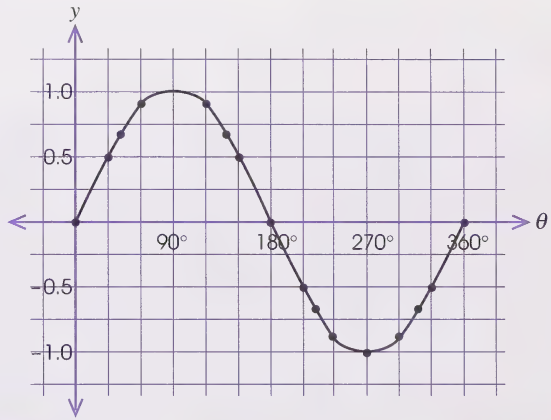
For example, $\frac{1}{2}$ reoccurs for 30° and 150°, and $-\frac{1}{2}$ reoccurs for 210° and 330°. A visual representation will give you a better idea of how all of the previous information fits together. (Can you predict the shape of the curve?)

Graph the information from the preceding charts.

Step 1: Let the x-axis represent the angles in degrees from 0° to 360°, and the y-axis represent the value of the function $y = \sin\ \theta$.

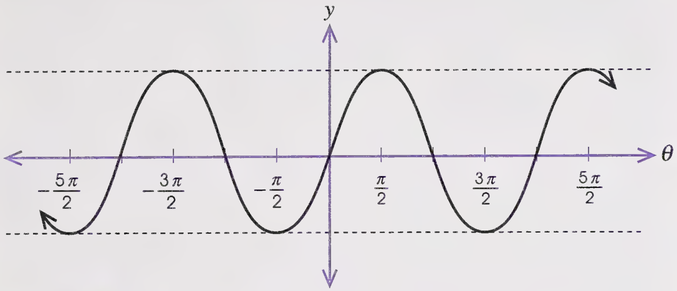
Step 2: Using the tables developed from the unit circle, plot the value of $y = \sin\ \theta$ for the special angles from 0° to 360°.

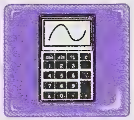
Step 3: Join the points with a smooth, best-fit curve.



Recall from your previous study of functions and relations that a **function** is a set of ordered pairs (x, y) in which each x is paired with exactly one value of y. The **domain** is the set of all x-values and the **range** is the set of all resulting y-values for a specific function. The relation $y = \sin\ \theta$ is a function, since any domain element (θ) has only one range element $(\sin\ \theta)$ associated with it for any ordered pair $(\theta, \sin\ \theta)$.

By observing the recurring nature of the sine curve, you can quickly sketch the graph of $y = \sin\ \theta$ for all values of θ by just using quadrantal angles as key points of reference. Remember, θ can be in either degrees or radians. In this case, radians will be used.



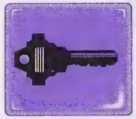


Use a graphing calculator or Zap-a-Graph™ to check the accuracy of the preceding sketch.

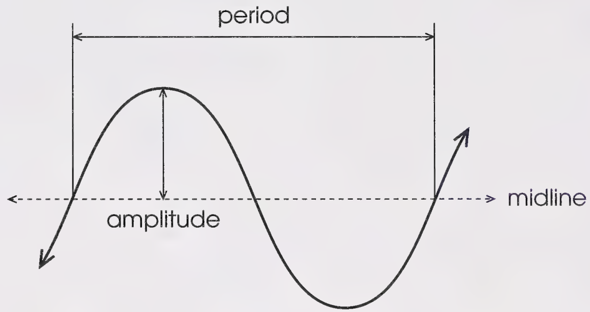
Note

- For the graphing calculator, the mode must be in radian measure.
- For Zap-a-Graph™, go to the Grid menu and change the scale for the major axis, x to π, by typing option P.

The function $y = \sin\ \theta$ is **periodic** since the values of y repeat themselves at equal intervals of θ. The **period** is defined as the horizontal distance from any point to the next point where the cycle begins again. For $y = \sin\ \theta$, the period is 2π rad or 360°.



A second term associated with the sine curve is **amplitude**. Amplitude is the maximum vertical distance of the curve from the midline. For $y = \sin\ \theta$, the amplitude is 1. Both amplitude and period are always positive quantities.



The domain of the function $y = \sin\ \theta$ is the description of all possible angles in either radians or degrees. Since $\sin\ \theta$ exists for all values of θ, the domain is $\{\theta | \theta \in R\}$. The range of the function $y = \sin\ \theta$ describes all values of $y = \sin\ \theta$. Since y has a maximum of 1 and a minimum of -1, the range is $\{y | -1 \le y \le 1\}$.

For $y = \sin\ \theta$,

Domain $= \{\theta | \theta \in R\}$ is read as follows:

The set of all values of θ such that θ is a member of the real numbers.

Range $= \{y | -1 \le y \le 1\}$ is read as follows:

The set of all values of y such that y is greater than or equal to -1 and less than or equal to 1.

Now, consider the graph of the trigonometric function $y = \tan\ \theta$. Use the unit circle to construct a table of values for special angles between $-\pi$ and 2π.

Recall: On the unit circle,

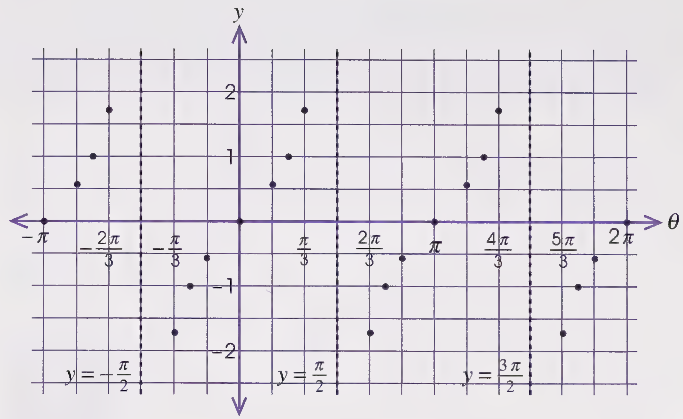
$$\tan\ \theta = \frac{y\text{-coordinate}}{x\text{-coordinate}}.$$

θ		$y = \tan\ \theta$	
degrees	**radians**	**exact**	**approximate**
-180	$-\pi$	0	0
-150	$\frac{-5\pi}{6}$	$\frac{\sqrt{3}}{3}$	0.6
-135	$\frac{-3\pi}{4}$	1	1
-120	$\frac{-2\pi}{3}$	$\sqrt{3}$	1.7
-90	$\frac{-\pi}{2}$	undefined	undefined
-60	$\frac{-\pi}{3}$	$-\sqrt{3}$	-1.7
-45	$\frac{-\pi}{4}$	-1	-1
-30	$\frac{-\pi}{6}$	$\frac{-\sqrt{3}}{3}$	-0.6
0	0	0	0
30	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$	0.6
45	$\frac{\pi}{4}$	1	1
60	$\frac{\pi}{3}$	$\sqrt{3}$	1.7
90	$\frac{\pi}{2}$	undefined	undefined

θ		$y = \tan\ \theta$	
degrees	radians	exact	approximate
120	$\frac{2\pi}{3}$	$-\sqrt{3}$	−1.7
135	$\frac{3\pi}{4}$	−1	−1
150	$\frac{5\pi}{6}$	$\frac{-1}{\sqrt{3}}$	−0.6
180	π	0	0
210	$\frac{7\pi}{6}$	$\frac{1}{\sqrt{3}}$	0.6
225	$\frac{5\pi}{4}$	1	1
240	$\frac{4\pi}{3}$	$\sqrt{3}$	1.7
270	$\frac{3\pi}{2}$	undefined	undefined
300	$\frac{5\pi}{3}$	$-\sqrt{3}$	−1.7
315	$\frac{7\pi}{4}$	−1	−1
330	$\frac{11\pi}{6}$	$\frac{-1}{\sqrt{3}}$	−0.6
360	2π	0	0

You can now plot the resulting ordered pairs if you let the x-axis represent values of θ and the y-axis represent the resulting values of $\tan\ \theta$.

Note: In $y = \tan\ \theta$, y is the vertical y-axis of the coordinate system used to graph the function and not the y-coordinate of a point on the unit circle.



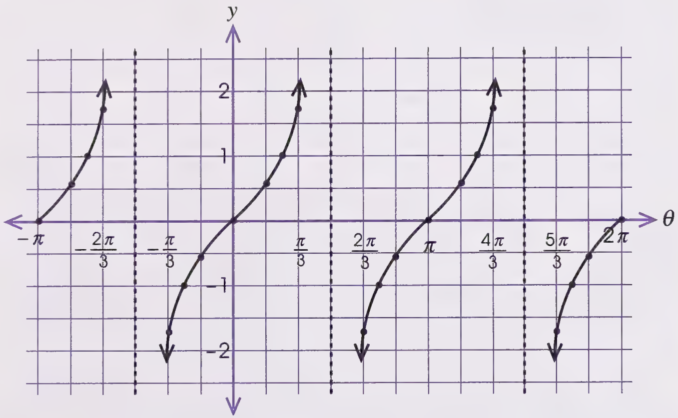
As you can see, $y = \tan\ \theta$ is undefined at $\theta = \frac{-\pi}{2}$ rad, $\frac{\pi}{2}$ rad, and $\frac{3\pi}{2}$ rad. You can conclude that $\tan\ \theta$ does not exist at these values.

The lines $\theta = \frac{-\pi}{2}$, $\frac{\pi}{2}$, and $\frac{3\pi}{2}$, and so on are vertical asymptotes.

The graphs for $\tan\ \theta$ extend indefinitely up and down, approaching these lines, but never touching them.

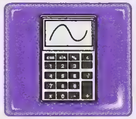
An **asymptote** is a line that a graph approaches, but never touches.

In order to be consistent with the plotted points, as θ approaches these values, the values of $y = \tan\ \theta$ become increasingly larger positive values or decreasingly smaller negative values. The graph of $y = \tan\ \theta$ can now be obtained by joining the plotted points with a smooth curve that approaches $\theta = \frac{-\pi}{2}$ rad, $\frac{\pi}{2}$ rad, and $\frac{3\pi}{2}$ rad without $\tan\ \theta$ existing at these angles.



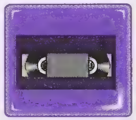
The period of the tangent function is π rad or 180°, since it takes a horizontal distance of π rad in order to complete one cycle. From this graph, you can see that $\theta \neq \frac{-\pi}{2}$, $\frac{\pi}{2}$, and $\frac{3\pi}{2}$. In general, for $y = \tan\ \theta$, the domain is $\left\{\theta \middle| \theta \neq \frac{\pi}{2} + n\pi,\ n \in I\right\}$. Since the graph extends vertically forever upwards and downwards, the range of $y = \tan\ \theta$ is $\left\{y \middle| y \in R\right\}$.

You should now be able to graph any of the six basic trigonometric functions and discuss their identifying features.



Use a graphing calculator or Zap-a-Graph™ to draw $y = \tan\ \theta$, and see how your graph compares with the preceding graph.

You will now be drawing some graphs and providing some information on each of them.



Before completing the questions that follow, watch the program *Graphing Trigonometric Functions I* from the *Graphing Mathematical Concepts* series, ACCESS Network. The video shows you how to sketch some of the graphs of the trigonometric functions. This video is available from the Learning Resources Distributing Centre.

1. Complete the following chart; then plot the ordered pairs $(\theta, \cos\,\theta)$ on a graph. Join the points with a best-fitting, smooth curve to obtain the graph of $y = \cos\,\theta$.

θ	**radians**	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
	degrees	0	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
$y = \cos\,\theta$	**exact**																	
	nearest tenth																	

2. Sketch the graph of $y = \cos\,\theta$ using quadrantal angles as a guide for $-\pi \leq \theta \leq 3\pi$.

3. For $y = \cos\,\theta$, state the following:

   a. domain **b.** range **c.** period **d.** amplitude

4. In summary, complete the following chart.

Function	Domain	Range	Period
$y = \sin\,\theta$			
$y = \cos\,\theta$			
$y = \tan\,\theta$			

Note: You may use a graphing calculator or Zap-a-Graph™ to compare the graphs with the ones you have drawn in questions 1 to 4.

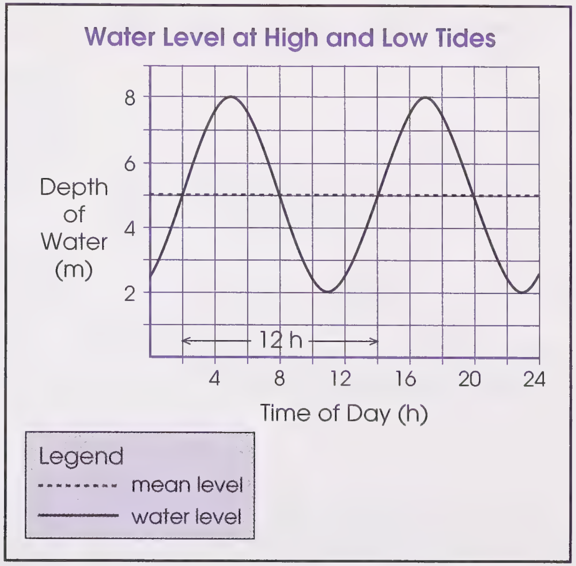
Check your answers by turning to the Appendix.

Your knowledge of the basic trigonometric graphs will help you to understand the terminology introduced in Activity 2.

Activity 2: Amplitude, Period, Phase Shift, Vertical Translation, and Range

Recall, from Section 3, that tides are caused by the gravitational pull of the moon and the sun, and that they are the periodic rise and fall of the water in the oceans.

The following graph shows the high and low tides during a typical day.



What was the depth of the water at 12.0 h on that day?

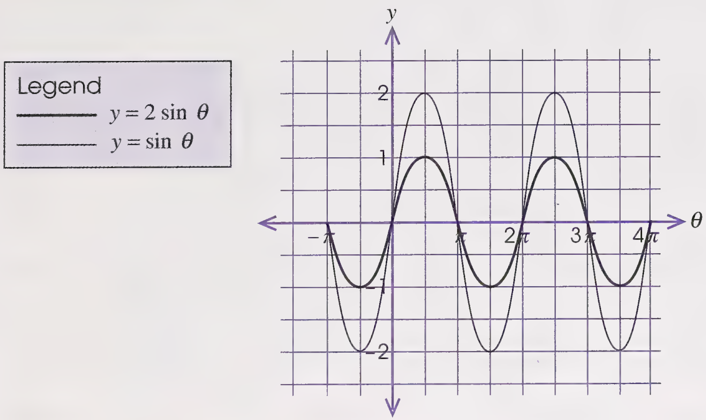
One way of answering this question would be to find the point corresponding to 12.0 h on the graph and reading the corresponding depth from the vertical axis. The approximation obtained in this way depends on the accuracy with which the graph was drawn and the scale that was used.

Perhaps a better approximation can be obtained if you are able to find an equation which defines depth as a function of time. The graph suggests the function is a sine (or cosine) function (hence the name **sinusoidal**), but it is obvious that the equation is not simply $y = \sin\ \theta$ (or $y = \cos\ \theta$).

To sketch graphs that are more complex than $y = \sin \theta$ or $y = \cos \theta$, the following terminology must be understood: amplitude, period, phase shift, and vertical translation.

Amplitude

The following are the graphs of $y = \sin \theta$ and $y = 2 \sin \theta$ for values of θ from $-\pi$ to 4π inclusive.



Study the graphs closely. You will notice that the zeros of $y = 2 \sin \theta$ are the same as the zeros of $y = \sin \theta$. The maximum and minimum for both graphs occur for the same values of θ. These maximum and minimum values for $y = 2 \sin \theta$ have a **magnification** of two as compared to $y = \sin \theta$. This magnification is caused by the factor 2 in the equation $y = 2 \sin \theta$, and you say that the **amplitude** is 2.

From the graph, the maximum values occur when $\theta = \frac{\pi}{2}$ and $\frac{5\pi}{2}$. The minimum values occur when $\theta = \frac{-\pi}{2}$, $\frac{3\pi}{2}$, and $\frac{7\pi}{2}$.

Similarly, you can find the amplitude of a cosine function. The amplitude of a sinusoidal function is one-half times the difference between the maximum and minimum values.

The following is a simple formula to find amplitude.

$$|a| = \frac{M - m}{2}$$

M is the maximum value, and m is the minimum value of the function.

For $y = 2 \sin \theta$, the maximum value is 2 and the minimum value is -2.

$$\begin{aligned}\therefore\ |a| &= \frac{M - m}{2} \\ &= \frac{2 - (-2)}{2} \\ &= \frac{2 + 2}{2} \\ &= 2\end{aligned}$$

The amplitude of the curve is 2.

Note

For $y = 2 \sin \theta$, the domain and range is as follows:

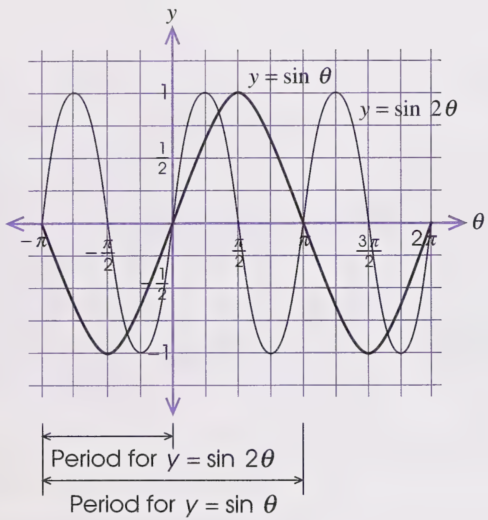
- Domain: $\{\theta | \theta \in R\}$
- Range: $\{y | -2 \le y \le 2\}$

In general, for $y = a \sin \theta$, the domain is R and the range is $\{y | -|a| \le y \le |a|\}$.

The amplitude of the function $y = a \sin \theta$ and $y = a \cos \theta$ is $|a|$.

Period

Look at the graphs of $y = \sin 2\theta$ and $y = \sin \theta$, where $-\pi \le \theta \le 2\pi$.



From this graph, one period of the graph of $y = \sin 2\theta$ is half as long as one period of the graph of $y = \sin \theta$.

Therefore, the function $y = \sin 2\theta$ has an amplitude of 1 and a period of $\frac{2\pi}{2}$ or π.

Thus, the graph $y = \sin \theta$ has been compressed horizontally by a factor of two to form the graph of $y = \sin 2\theta$.

In the same way, the period of $y = \cos b\theta$ can be found by dividing 2π by b.

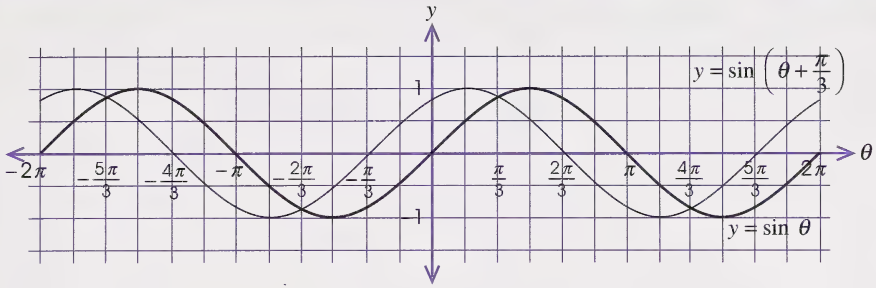
The functions $y = \sin b\theta$ and $y = \cos b\theta$ have the period $p = \frac{2\pi}{b}$, $b \neq 0$.

$|b|$ is a value which allows you to determine the period of the trigonometric function.

Phase Shift

The following table of values has been used to draw the graphs of $y = \sin\left(\theta + \frac{\pi}{3}\right)$ and $y = \sin \theta$, where $-2\pi \le \theta \le 2\pi$. The graphs are drawn from -2π to 2π but the table of values is from $-\pi$ to π which is sufficient since the graphs repeat their cycles.

θ	$-\pi$	$\frac{-5\pi}{6}$	$\frac{-2\pi}{3}$	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$y = \sin\theta$	0	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0
$y = \sin\left(\theta + \frac{\pi}{3}\right)$	$\frac{-\sqrt{3}}{2}$	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$\frac{-1}{2}$	$\frac{-\sqrt{3}}{2}$



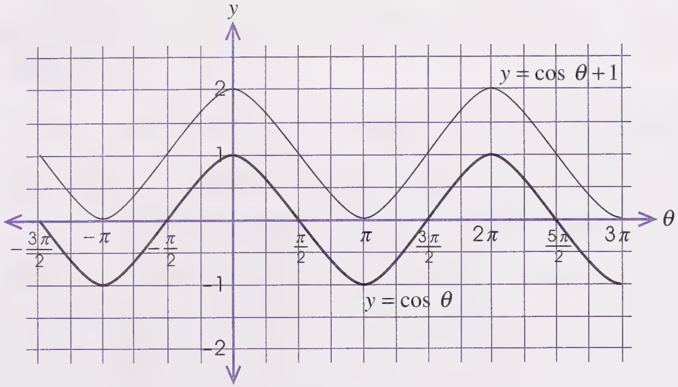
Notice that the graph $y = \sin\left(\theta + \frac{\pi}{3}\right)$ is similar to the graph $y = \sin\theta$, but is shifted to the left. The table of values also illustrates this phase shift. In the table of values, find θ equal to $\frac{2\pi}{3}$. Then notice all the function values starting at this point and going to the left for the graph $y = \sin\left(\theta + \frac{\pi}{3}\right)$. The values are 0, $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, 1, $\frac{\sqrt{3}}{2}$, and so on. Now, in the table of values, find θ equal to π. Then examine all the function values starting at this point and going to the left for the graph $y = \sin\theta$. The values are 0, $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, 1, $\frac{\sqrt{3}}{2}$, and so on. Thus, the graph of $y = \sin\left(\theta + \frac{\pi}{3}\right)$ is identical to the graph of $y = \sin\theta$, but is shifted $\frac{\pi}{3}$ units to the left.

In trigonometry, this shifting is called the **phase shift** of the function.

The phase angle for $y = \sin(\theta + c)$ and $y = \cos(\theta + c)$ is the constant angle c. The phase shift is c units to the right if c is negative, and c units to the left if c is positive.

Vertical Translation

The following shows the graphs of $y = \cos\ \theta + 1$ and $y = \cos\ \theta$, where $-\frac{3\pi}{2} \leq \theta \leq 3\pi$.



Note that the graph of $y = \cos\ \theta + 1$ is the same as the graph of $y = \cos\ \theta$ shifted vertically upward by 1 unit.

The **vertical translation** for $y = \cos\ \theta + d$ and $y = \sin\ \theta + d$ is d units upward if d is positive, and d units downwards if d is negative.

Thus, for $y = a \sin\left[b(\theta + c)\right] + d$ or $y = a \cos\left[b(\theta + c)\right] + d$, $|a|$ is the amplitude, $|b|$ is a value which allows you to determine the period of the function using $p = \frac{2\pi}{b}$, c is the phase shift or horizontal translation (slide), and d is the vertical translation (slide).

If you refer back to the graph of Water Level at High and Low Tides at the beginning of this Activity, you will observe that the amplitude of the tidal depths is 3. By calculation, since the maximum value is 8 and the minimum value is 2, the amplitude is $a = \frac{1}{2}(8 - 2) = 3$. You will also notice that the period of the tidal depth function is 12.

Now look at some examples.

Example 1

The equation of a cosine graph is $y = 3\cos\ \frac{1}{2}\theta$. Find the amplitude and the period.

Solution

$$3\cos\ \frac{1}{2}\theta = a\cos\ \frac{1}{2}\theta$$

$$\therefore\ |a| = 3$$

The amplitude is 3.

$$\begin{aligned} p &= \frac{2\pi}{b} \\ &= \frac{2\pi}{\frac{1}{2}} \\ &= 4\pi \end{aligned}$$

The period is 4π.

Example 2

If $y=2\sin(3\theta+\pi)-1.5$, give the phase shift and vertical translation.

Solution

Rewrite the expression as a general equation.

$$y=a\sin\left[b(\theta+c)\right]+d$$

$$\therefore\ 2\sin(3\theta+\pi)-1.5=2\sin\left[3\left(\theta+\frac{\pi}{3}\right)\right]-1.5$$

The graph of $y=\sin\theta$ is shifted $\frac{\pi}{3}$ units to the left.
The graph of $y=\sin\theta$ is shifted 1.5 units downward.

Example 3

Find the amplitude, period, phase shift, and vertical translation for a curve represented by $y=-\frac{1}{2}\cos\left(\frac{1}{4}\theta-\frac{\pi}{12}\right)+3$.

Solution

Find the general equation.

$$\begin{aligned} y&=-\frac{1}{2}\cos\left(\frac{1}{4}\theta-\frac{\pi}{12}\right)+3\\ &=-\frac{1}{2}\cos\left[\frac{1}{4}\left(\theta-\frac{\pi}{3}\right)\right]+3 \end{aligned}$$

$$\begin{aligned} |a|&=\left|-\frac{1}{2}\right|\\ &=\frac{1}{2} \end{aligned}$$

$$\begin{aligned} p&=\frac{2\pi}{b}\\ &=\frac{2\pi}{\frac{1}{4}}\\ &=8\pi \end{aligned}$$

Therefore, the amplitude is $\frac{1}{2}$; the period is 8π; the phase shift is $\frac{\pi}{3}$ to the right; and the vertical translation is 3 units upward.

1. State the amplitude and period for each of the following:

   a. $y=0.5\sin 4\theta$ **b.** $y=\sin\frac{\theta}{5}$

   c. $y=-4\cos\frac{2m}{3}$ **d.** $f(x)=3\cos 4x$

   e. $f(\theta)=-\sin(-2\theta)$ **f.** $y=2\cos(2\theta+\pi)$

2. Give the phase shift and vertical translation for the following:

   a. $y=\frac{1}{2}\sin\theta-2.5$

   b. $y=\cos\left(2\theta-\frac{\pi}{6}\right)$

   c. $f(\theta)=3\sin(4\theta+3\pi)+2$

   d. $y=12\sin\left(t+\frac{\pi}{4}\right)+7$

   e. $f(\theta)=2\cos(3\theta-\pi)+3$

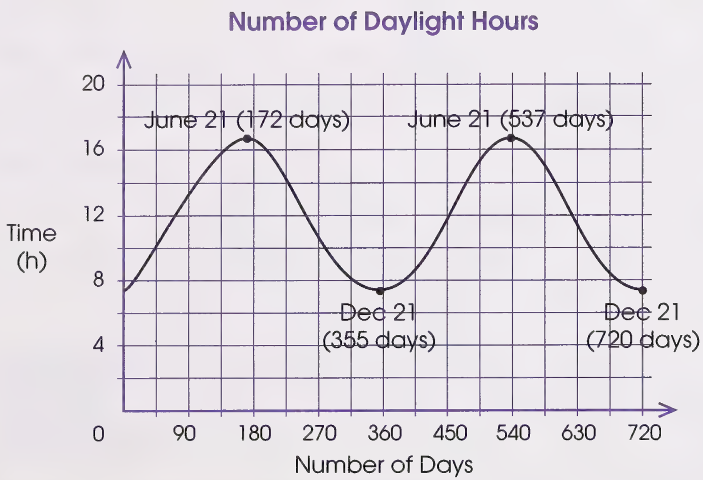
   f. $f(x)=-\sin\left(x+\frac{\pi}{4}\right)+5$

3. State the amplitude, period, phase shift, and vertical translation for the following functions:

a. $y = 3\cos(2\theta - \pi) - 3.5$

b. $f(x) = -2\sin\left(\frac{1}{2}\theta + \frac{\pi}{2}\right) + 2$

4. In summer, the hours of daylight are longer than in winter. The graph shows the hours of daylight for a two-year period in Edmonton.



a. Estimate the period for the function illustrated in the graph.
b. Estimate the amplitude for the graph.

You may wish to check your answers by using Zap-a-Graph™. The transformations are given under "Analyze" in the Options menu.

Check your answers by turning to the Appendix.

Example 4

Write an equation for the cosine function given that the amplitude is -2 and the period is π.

Solution

Since there is no phase shift and vertical translation, the form is $y = a\cos b\theta$.

Since the amplitude is -2, then $a = -2$.

$$p = \frac{2\pi}{b}$$
$$\pi = \frac{2\pi}{b}$$
$$b = \frac{2\cancel{\pi}}{\cancel{\pi}}$$
$$= 2$$

Therefore, $y = a\cos b\theta$ becomes $y = -2\cos 2\theta$.

Example 5

Write an equation for the sine function having an amplitude of 3, a period of $\frac{\pi}{2}$, a phase shift of $\frac{\pi}{4}$ to the left, and a vertical translation of -2.

Solution

The form is $y = a\sin\left[b(\theta + c)\right] + d$.

$$a = 3 \qquad c = \frac{\pi}{4} \qquad d = -2$$

$$\begin{aligned} p &= \frac{2\pi}{b} \\ \frac{\pi}{2} &= \frac{2\pi}{b} \\ b &= \frac{4\cancel{\pi}}{\cancel{\pi}} \\ &= 4 \end{aligned}$$

Therefore, $y = a\sin\left[b(\theta + c)\right] + d$ becomes $y = 3\sin 4\left(\theta + \frac{\pi}{4}\right) - 2$.

5. Write an equation for the sine function given the following:

a. an amplitude of 2 and a period of $-\pi$

b. an amplitude of -3, a period of $\frac{2\pi}{3}$, and a phase shift of $\frac{\pi}{2}$ to the left

Note: If the amplitude and the period are given and the equation must be determined, then use the amplitude and period with the given signs. Do not use the absolute values of the amplitude and period when writing equations.

6. Write an equation for the cosine function having the following properties:

a. an amplitude of 1, a period of 2π, and a phase shift of $\frac{\pi}{3}$ to the right

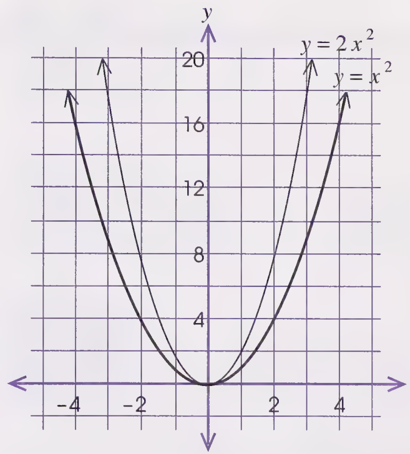
b. an amplitude of -5, a period of $\frac{\pi}{3}$, a phase shift of $\frac{\pi}{4}$ to the left, and a vertical translation of -2

Check your answers by turning to the Appendix.

The terminology in this activity will help you to investigate the effects of the parameters *a*, *b*, *c*, and *d* on the graphs of trigonometric functions.

Activity 3: Graphing Functions of the Form $y = a \sin [b(\theta + c)] + d$ or $y = a \cos [b(\theta + c)] + d$

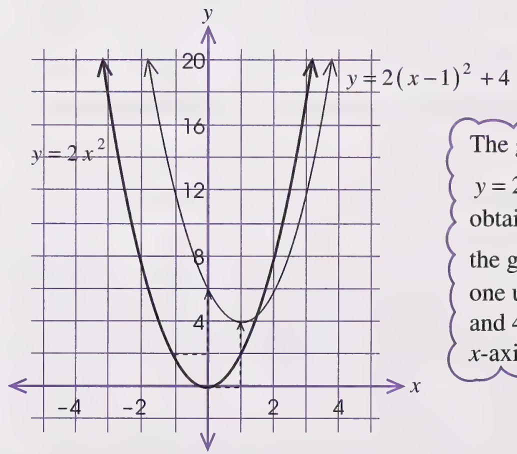
Now, you will investigate transformations of the graphs of $y = \sin \theta$ and $y = \cos \theta$. The relationship between the graphs of $y = x^2$ and $y = 2(x-1)^2 + 4$ is shown to the right. The value in front of the parentheses (in this case, positive 2) gives the direction of the opening and the relative steepness of the graph. If you consider the form $y = a(x-h)^2 + k$, the value of h (in this case, 1) and the value of k (in this case, 4) are the x- and y-coordinates of the vertex respectively. So, the vertex is translated from $(0,0)$ to $(1,4)$. In fact, each point of the graph of $y = 2(x-1)^2 + 4$ is 1 unit to the right and 4 units up from a corresponding point on $y = 2x^2$.



The coordinates of the vertex for $y = 2x^2$ are $(0,0)$.

To show the relationship, do the following:

- Draw the graph of $y = x^2$.
- Overlay the graph of $y = 2x^2$. (Every ordinate is twice the corresponding ordinate of $y = x^2$.)
- Overlay the graph of $y = 2(x-1)^2 + 4$.



The graph of $y = 2(x-1)^2 + 4$ is obtained by shifting the graph of $y = 2x^2$ one unit to the right and 4 units above the x-axis.

You can use these ideas in addition to what you learned in Section 4: Activity 2 to draw the graphs of functions such as $y = 2\sin\left(\theta - \frac{\pi}{4}\right) + 1$.

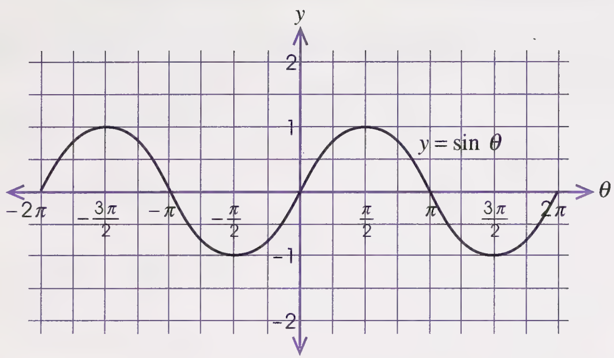
You will do this in the following examples.

Example 1

Draw the graph of $y = 2\sin\theta$.

Solution

Draw the graph of $y = \sin\theta$, where $-2\pi \le \theta \le 2\pi$.



The amplitude (vertical stretch) of $y = 2\sin\theta$ is 2.

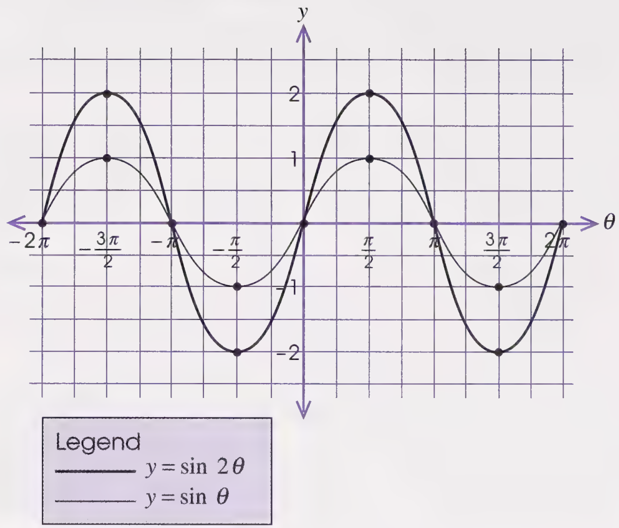
The period for $\sin\theta$ and $2\sin\theta$ is 2π.

Each ordinate of $y = \sin\theta$ is doubled to get the ordinate of $y = 2\sin\theta$.

The following table shows these values.

θ	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \sin\theta$	0	1	0	−1	0	1	0	−1	0
$y = 2\sin\theta$	0	2	0	−2	0	2	0	−2	0

Now draw the graph of $y = 2\sin\theta$.



Example 2

Draw the graph of $y=\frac{1}{2}\sin\left(2\theta+\frac{\pi}{2}\right)$, where $-2\pi\le\theta\le2\pi$.

Solution

To obtain the phase shift, rewrite the expression in the general form $y=a\sin\left[b(\theta+c)\right]+d$.

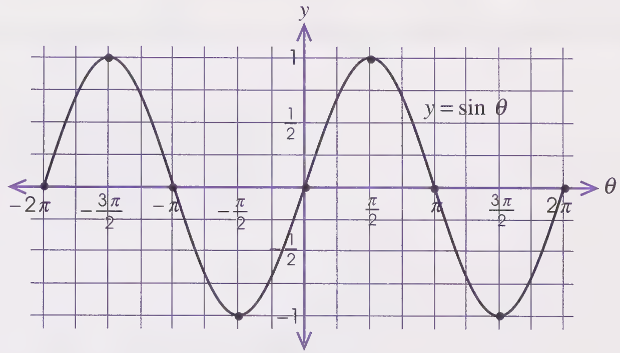
$$\begin{aligned} y&=\tfrac{1}{2}\sin\left(2\theta+\tfrac{\pi}{2}\right)\\ &=\tfrac{1}{2}\sin\left[2\left(\theta+\tfrac{\pi}{4}\right)\right]\end{aligned}$$

Use the following steps to develop this graph.

Step 1: Graph $y=\sin\theta$, where $-2\pi\le\theta\le2\pi$.

Construct a table of values for $y=\sin\theta$, and draw the graph.

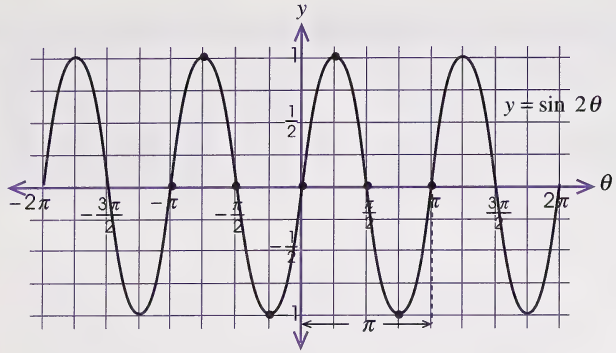
$\boldsymbol{\theta}$	-2π	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\boldsymbol{y=\sin\theta}$	0	1	0	−1	0	1	0	−1	0



The period for $y=\sin\theta$ is 2π. Two cycles (periods) are shown in this graph: -2π to 0 and 0 to 2π.

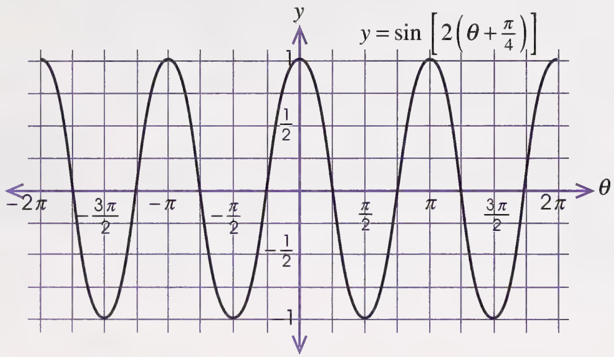
Step 2: Remember that the period for $y=\sin(2\theta)$ is $\frac{2\pi}{2}=\pi$. Thus, the graph repeats itself every π units. The horizontal stretch factor is $\frac{1}{2}$, which results in a compression of the $y=\sin\theta$ curve. Construct a table of values for $y=\sin2\theta$, and draw the graph.

$\boldsymbol{\theta}$	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$\boldsymbol{y=\sin2\theta}$	0	1	0	−1	0	1	0	−1	0

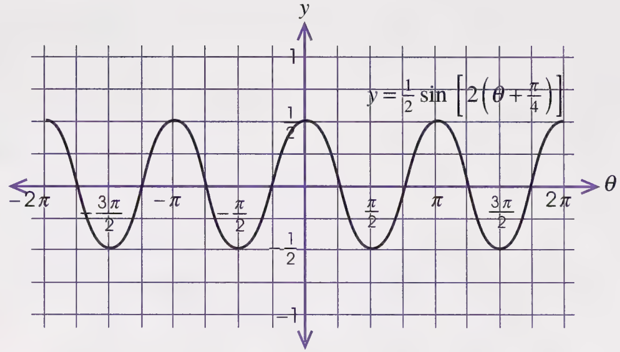


Four cycles are shown in this graph: -2π to $-\pi$, $-\pi$ to 0, 0 to π, and π to 2π.

Step 3: A phase shift of $\frac{\pi}{4}$ to the left is applied to the curve $y = \sin 2\theta$ in order to obtain the curve $y = \sin\left[2\left(\theta + \frac{\pi}{4}\right)\right]$.

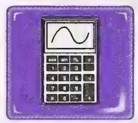


Step 4: A vertical stretch of factor $\frac{1}{2}$ (that is, amplitude is $\frac{1}{2}$) is applied to $y = \sin\left[2\left(\theta + \frac{\pi}{4}\right)\right]$ to obtain the graph of $y = \frac{1}{2}\sin\left[2\left(\theta + \frac{\pi}{4}\right)\right]$.



This is the required graph for $y = \frac{1}{2}\sin\left(2\theta + \frac{\pi}{2}\right)$ in the interval $-2\pi \le \theta \le 2\pi$.

Note: The graph of $y = \frac{1}{2}\sin\left[\left(2\theta + \frac{\pi}{2}\right)\right]$ is also the graph of $y = \frac{1}{2}\sin\left[2\left(\theta + \frac{\pi}{4}\right)\right]$.



Draw this graph with a graphing calculator and compare the graphs. The CASIO *fx-7700G* graphing calculator accepts the form $y = \frac{1}{2}\sin\left[\left(2\theta + \frac{\pi}{2}\right)\right]$.

Example 3

Discuss the transformation of the graph of $y = \sin\ \theta$ that yields

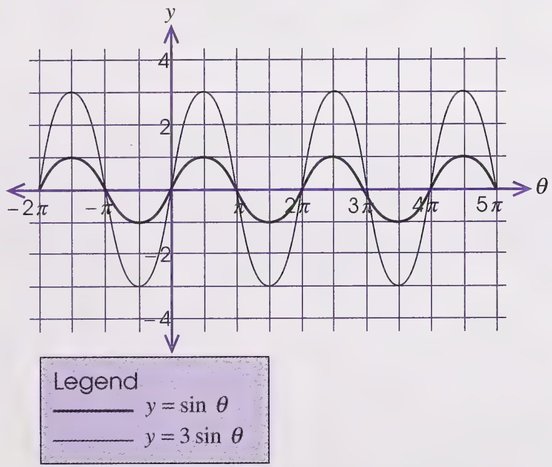
$y = 3\sin\left[\frac{1}{2}\left(\theta + \frac{\pi}{2}\right)\right] - 1.$

Graph $y = 3\sin\left[\frac{1}{2}\left(\theta + \frac{\pi}{2}\right)\right] - 1$, where $-2\pi \leq \theta \leq 5\pi$, and give its range.

Solution

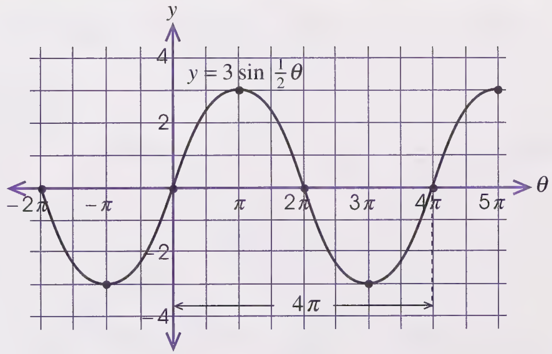
Since $|a| = 3$, then the amplitude is 3.

The graph of $y = \sin\ \theta$ has a range of $-1 \leq y \leq 1$. A vertical stretch of factor 3 changes the range to $-3 \leq y \leq 3$.



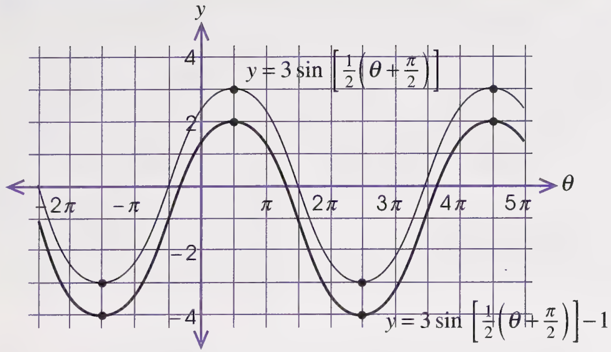
Since $b = \frac{1}{2}$, then only $\frac{1}{2}$ of one cycle will be completed within the period of $y = \sin\ \theta$. It will take 4π to complete one full cycle. The period of this function is $\frac{2\pi}{\frac{1}{2}} = 4\pi$.

θ	-2π	$-\pi$	0	π	2π	3π	4π	5π
$3\sin\frac{1}{2}\theta$	0	−3	0	3	0	−3	0	3



Since $c = \frac{\pi}{2}$ and $d = -1$, the curve has been shifted $\frac{\pi}{2}$ units to the left and 1 unit down. This downward shift makes the minimum value -4 and the maximum value 2. As a result, the range is $-4 \leq y \leq 2$.

θ	$-\frac{3\pi}{2}$	$-\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	$\frac{5\pi}{2}$	$\frac{7\pi}{2}$	$\frac{9\pi}{2}$
$3\sin\frac{1}{2}\left(\theta+\frac{\pi}{2}\right)$	−3	0	3	0	−3	0	3
$3\sin\frac{1}{2}\left(\theta+\frac{\pi}{2}\right)-1$	−4	−1	2	−1	−4	−1	2



The range for this graph is $\{y \mid -4 \le y \le 2\}$.

Use a graphing calculator (or Zap-a-Graph™) to draw the graphs in Example 3. Compare the graphs.

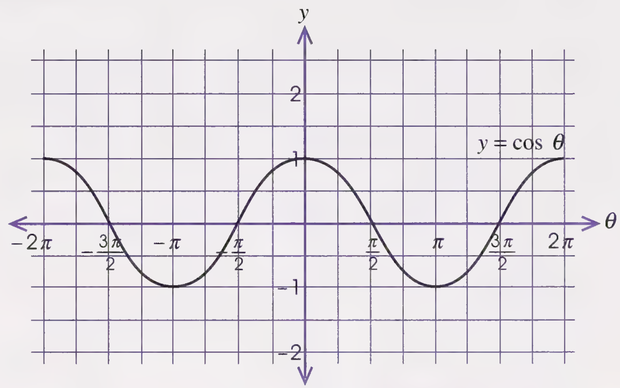
Now that you know how to sketch a sine graph, you will use this knowledge to sketch a cosine graph. The cosine graph, $y=a\cos\left[b(\theta+c)\right]+d$ also undergoes four transformations. The next example shows the various steps in the development of this type of graph.

Example 4

Draw the graph of $y=1.5\cos\left[2\left(\theta+\frac{\pi}{4}\right)\right]+0.5$, where $-2\pi \le \theta \le 2\pi$.

Solution

Step 1: Draw the graph of $y=\cos\theta$, where $-2\pi \le \theta \le 2\pi$.

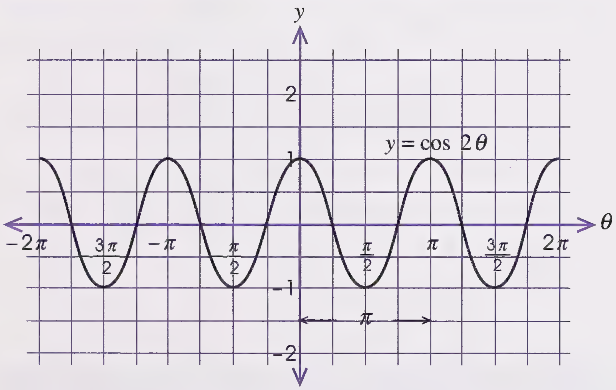


The range for $y=\cos\theta$ is $\{y \mid -1 \le y \le 1\}$.

Step 2: The period for $y = \cos\ 2\theta$ is $\frac{2\pi}{2} = \pi$. Thus, the graph repeats itself every π units. The horizontal stretch factor is $\frac{1}{2}$, which results in a compression of the $y = \cos\ \theta$ curve. The following is a table of values that will help you to construct the graph.

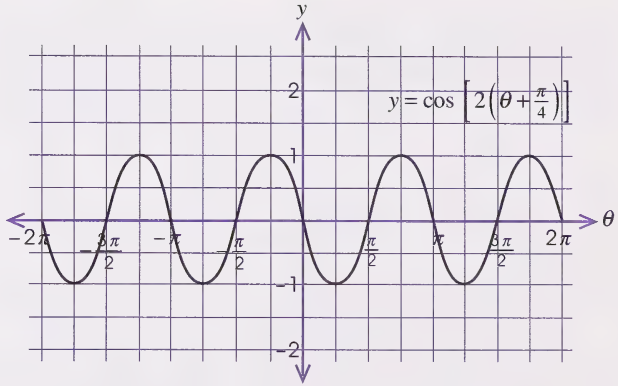
θ	-2π	$-\frac{7\pi}{4}$	$-\frac{3\pi}{2}$	$-\frac{5\pi}{4}$	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0
$\cos\ 2\theta$	1	0	−1	0	1	0	−1	0	1

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\cos\ 2\theta$	0	−1	0	1	0	−1	0	1

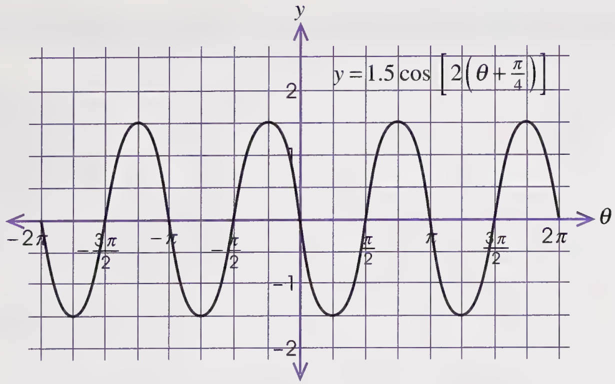


The range for $y = \cos\ 2\theta$ is $\{y \mid -1 \le y \le 1\}$.

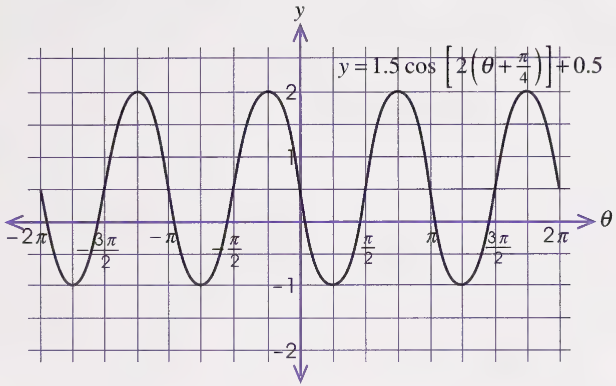
Step 3: To obtain the graph of $y = \cos\left[2\left(\theta + \frac{\pi}{4}\right)\right]$, perform a phase shift on the graph of $y = \cos\ 2\theta$. For the variable expression $\left(\theta + \frac{\pi}{4}\right)$, the phase shift is $\frac{\pi}{4}$ to the left.



Step 4: The amplitude of $1.5\cos\ 2\left(\theta + \frac{\pi}{4}\right)$ is 1.5. So, perform a vertical stretch of factor 1.5 on $y = \cos\ 2\left(\theta + \frac{\pi}{4}\right)$.



Step 5: Since $d = 0.5$, the graph of $y = 1.5\cos 2\left(\theta + \frac{\pi}{4}\right)$ is translated up 0.5 units. Examine the graph; then check with the table of values as an assurance.



θ	$-\frac{7\pi}{4}$	$-\frac{5\pi}{4}$	$-\frac{3\pi}{4}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$
y	−1	2	−1	2	0.5	−1	2	−1	2

The range for this graph is $\{y \mid -1 \le y \le 2\}$.

You may use Zap-a-Graph™ to draw the curves in Example 4. Compare the graphs.

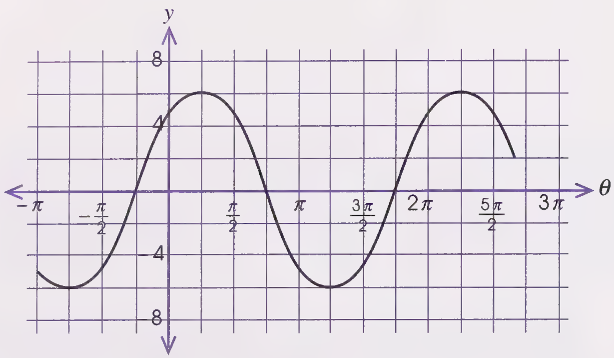
Before completing the questions that follow, watch the program entitled *Graphing Trigonometric Functions II* from the *Graphing Mathematical Concepts* series, ACCESS Network. This video shows the transformations of the graph $y = a\sin\left[b(\theta + c)\right] + d$. This video is available from the Learning Resources Distributing Centre.

1. For the graph $y = a\cos\left[b(\theta + c)\right] + d$, explain the role of a.

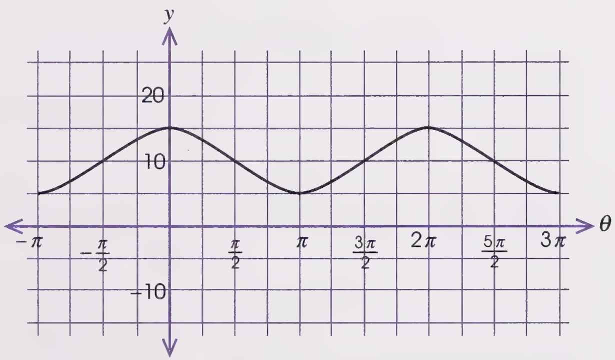
2. Examine Graph A, Graph B, and Graph C. For each graph, state the following:

   a. amplitude
   b. phase shift as compared to $y = \sin\theta$ or $y = \cos\theta$
   c. period
   d. maximum and minimum values
   e. vertical translation
   f. range of the function

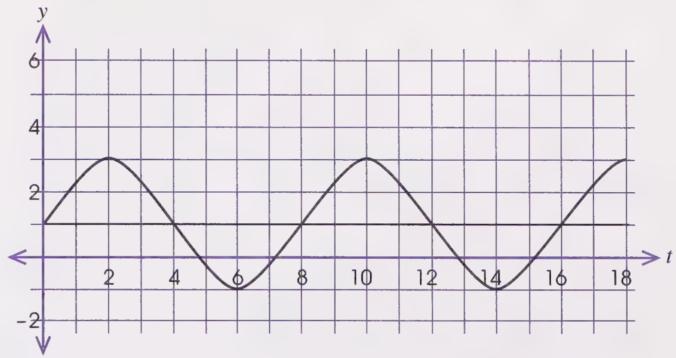
Graph A



Graph B



Graph C



- $|a| = \frac{M - m}{2}$
- $p = \frac{2\pi}{b}$
- $d = \frac{M + m}{2}$

3. **a.** Write an equation to represent the sine and cosine functions for Graph A and Graph B in question 2.

b. Write an equation to represent the sine function for Graph C in question 2.

4. Explain the meaning of c in $y = a \sin\left[b(\theta + c)\right] + d$.

5. Graph the function $y = 2\sin\left[2\left(\theta - \frac{\pi}{4}\right)\right] + 3$, and state the range.

6. If $y = 3\cos(2x - 120°) - 1$, determine the phase shift and period; then draw the graph.

Check your answers by turning to the Appendix.

The following example is an application of a trigonometric graph.

Example 5

A windmill with a radius of 2.0 m makes one complete revolution every 8 s. The vanes of the windmill are constructed in such a way that the windmill rotates counterclockwise. The bottom of the wheel is 10 m above the ground. Draw a diagram to show how a point on the rim of the wheel above the ground varies with time. Then find an equation of the graph.

Solution

Take a point (A) 12 m from the ground on the rim of the windmill. This is the initial point for a basic graph.

The highest point of the wheel is 14 m above the ground, and the lowest point is 10 m above the ground. These are the maximum and minimum values respectively.

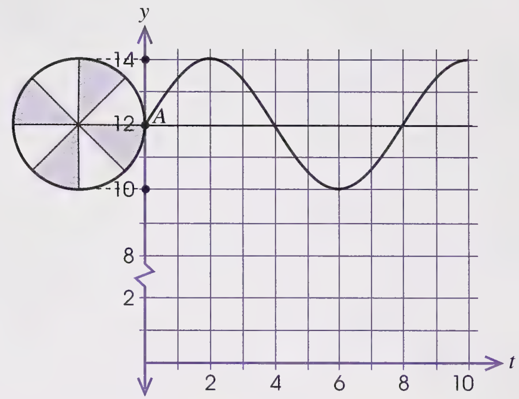
The radius is 2.0 m.

$$\therefore \text{ maximum height} = 12 + 2 = 14 \text{ m}$$

$$\text{minimum height} = 12 - 2 = 10 \text{ m}$$

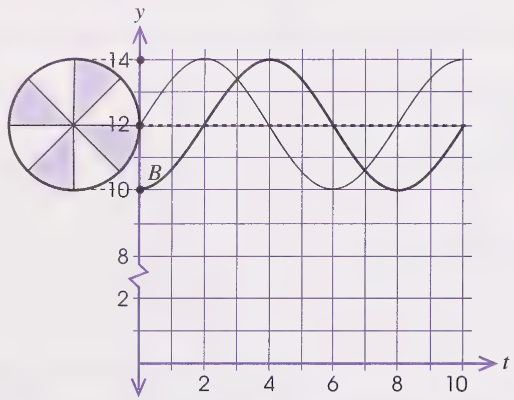
The period is 8 s.

Draw the basic sine curve. The axis of symmetry for this graph is $y = 12$ m.



Now take a point (B) as the initial point 10 m above the ground. This is the lowest point on the rim of the windmill. The maximum value is 14 m. Using the period 8 s, draw the second graph.

t	0	2	4	6	8
y	10	12	14	12	10



The general sine function is $y = a \sin\left[b(t+c)\right] + d$. The values for a, b, c, and d must be obtained before the equation of the graph can be determined.

From the graph,

$$|a| = \frac{M-m}{2} = \frac{14-10}{2} = \frac{4}{2} = 2$$

$$p = \frac{2\pi}{b}$$
$$8 = \frac{2\pi}{b}$$
$$b = \frac{\pi}{4}$$

The phase shift is 2 units to the right; therefore, $c = -2$.

The vertical translation is $d = \frac{M+m}{2}$

$$= \frac{14+10}{2} = 12$$

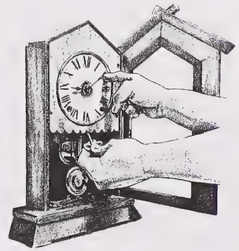
Therefore, the equation of the graph is $y = 2 \sin\left[\frac{\pi}{4}(t-2)\right] + 12$.

7. A Ferris wheel has a diameter of 12 m. It revolves in a counterclockwise direction and makes one complete revolution every 16 s.

a. Point B is on the Ferris wheel at the 6 o'clock position. Draw a graph to show the different positions of Point B if the bottom of the wheel is 1 m above the ground.

b. Write an equation for the graph.

8. For a pendulum, the angle θ is given by the equation $\theta = 2 \sin\left(\frac{1}{2}\pi t\right)$, where t is time (in seconds).



a. Draw the graph of the given function.

b. Find the values of t when $\theta = 0$.

Check your answers by turning to the Appendix.

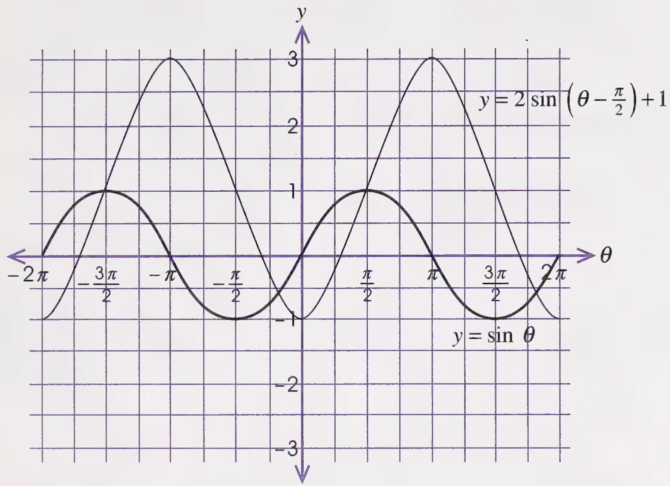
The following example shows the effects of the parameters on a given graph.

Example 6

Use the graph of $y = \sin\,\theta$ and $y = 2\sin\left(\theta - \frac{\pi}{2}\right) + 1$ to describe the effects of the parameters 2, $\frac{-\pi}{2}$, and 1 on the graph of $y = \sin\,\theta$.

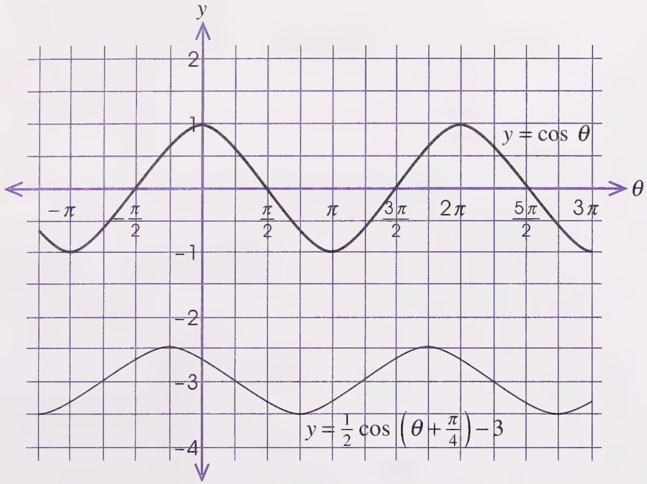
Solution

The graphs of $y = \sin\,\theta$ and $y = 2\sin\left(\theta - \frac{\pi}{2}\right) + 1$ are shown as follows:



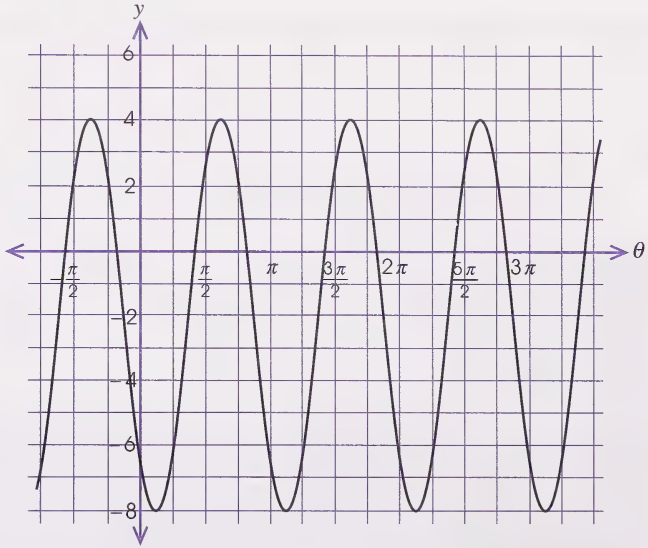
Parameter	Description	Effects on the Graph of $y = \sin\,\theta$
2	amplitude	changes the distance between the maximum and minimum values from 2 to 4, or, the distance from the midline to the maximum and minimum increases from 1 to 2
$-\frac{\pi}{2}$	horizontal translation	moves the graph of $y = \sin\,\theta$ to the right $\frac{\pi}{2}$ units
1	vertical translation	moves the midline of $2\sin\left(\theta - \frac{\pi}{2}\right)$ up one unit

9. The graphs of $y = \cos\theta$ and $y = \frac{1}{2}\cos\left(\theta + \frac{\pi}{4}\right) - 3$ are shown.



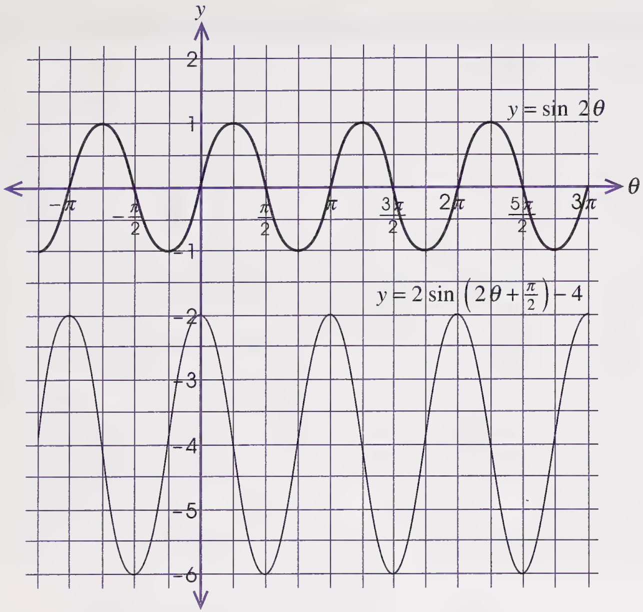
Describe the effects of the parameters $\frac{1}{2}$, $\frac{\pi}{4}$, and -3 on the graph of $y = \cos\theta$.

10.



The preceding graph is a trigonometric function in the form $y = a\sin\left[b(\theta + c)\right] + d$. Eileen determined from the graph that the amplitude was 12, the period was $\frac{4}{5}\pi$, and the range was R. Give a reason for each answer where Eileen went wrong and give the correct answers. Calculate the value of d.

11. Study the graphs of $y = \sin 2\theta$ and $y = 2\sin\left(2\theta + \frac{\pi}{2}\right) - 4$.



Describe the effects of the parameters 2, $\frac{\pi}{4}$, and -4 on the graph of $y = \sin 2\theta$.

Check your answers by turning to the Appendix.

You should be comfortable in recognizing the effects of the parameters on the graphs of the trigonometric functions discussed in this activity.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

The following is a summary of what you have studied from the three types of graphs of trigonometric functions.

sin *θ*

- The graph goes through the origin.
- The domain is *R*.
- The range is $\{y \mid -1 \le y \le 1\}$.
- The period is 2π rad.

cos *θ*

- The graph does not go through the origin.
- The domain is *R*.
- The range is $\{y \mid -1 \le y \le 1\}$.
- The period is 2π rad.

tan *θ*

- The graph lies within vertical asymptotes.
- The domain is $\{\theta \mid \theta \ne \frac{\pi}{2} + n\pi,\ n \in I\}$.
- The range is *R*.
- The period is π rad.

1. Give the meaning of vertical asymptote and period.

2. Define domain and range.

Check your answers by turning to the Appendix.

The following is a summary of what you studied in Activities 2 and 3:

- The amplitude of the function $y = a \sin \theta$ is $|a|$. Amplitude is the magnification or vertical stretch of the graph of $y = \sin \theta$ by a factor $|a|$.
- The period of the function $y = \sin k\theta$ is $\frac{2\pi}{k}$, where $k > 0$. (**Note:** k is used instead of b.)
- The phase shift for $y = \sin(\theta + c)$ is c units to the right if c is negative, and c units to the left if c is positive.
- In the function $y = \sin \theta + d$, the parameter d is the vertical translation. The graph is shifted d units upward if d is positive, and d units downward if d is negative.

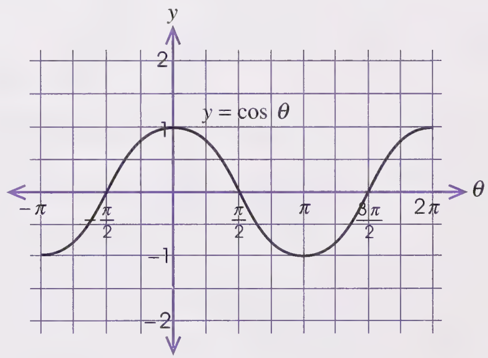
3. For the sinusoidal function $y = a \cos[b(\theta + c)]$, where $a > 0$, give the following values:

   a. period **b.** phase shift
   c. vertical translation **d.** amplitude
   e. range

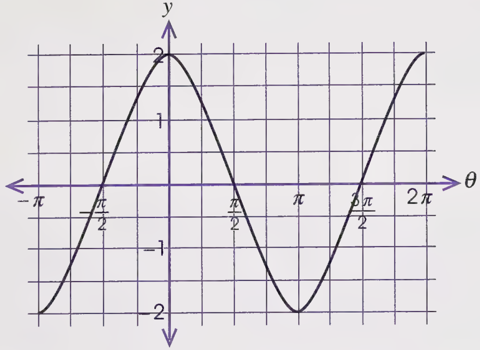
4. If $y = 2 \cos\left(\frac{\theta}{3}\right)$, state the following:

   a. amplitude **b.** period
   c. phase shift

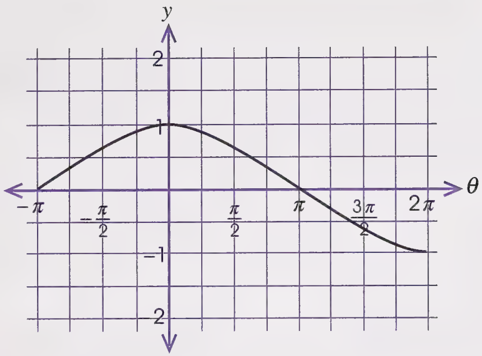
5. Use the following graph to do questions 5.a. and 5.b.



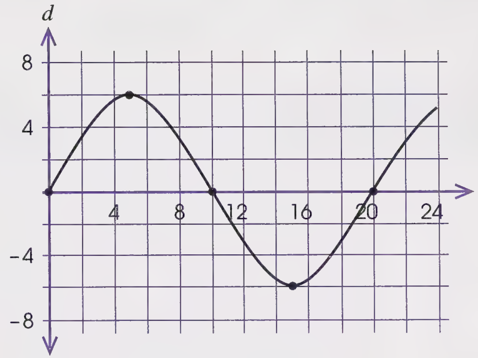
a. Find the defining equation.



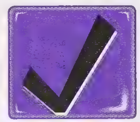
b. Find the amplitude, the period, and the defining equation.



6. The following graph describes the rise and fall of an ocean tide at a particular station, where d is height (in metres) and t is time (in hours).



a. What is the maximum value?
b. Give the period.
c. For what value of t is the height a minimum?
d. What are the t-intercepts?



Check your answers by turning to the Appendix.

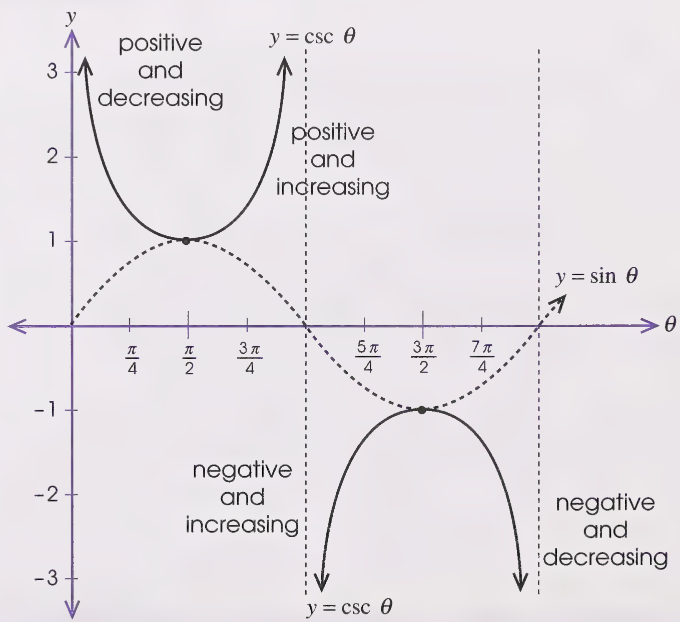
Enrichment

You can graph the function $y = \csc\,\theta$ utilizing the reciprocal relationship $\csc\,\theta = \frac{1}{\sin\,\theta}$ instead of setting up a table of values.

θ	$\sin\,\theta$	$\csc\,\theta$
0	0	undefined
$0 < \theta < \frac{\pi}{2}$	positive and increasing	positive and decreasing
$\frac{\pi}{2}$	1	1
$\frac{\pi}{2} < \theta < \pi$	positive and decreasing	positive and increasing
π	0	undefined
$\pi < \theta < \frac{3\pi}{2}$	negative and decreasing	negative and increasing
$\frac{3\pi}{2}$	-1	-1
$\frac{3\pi}{2} < \theta < 2\pi$	negative and increasing	negative and decreasing
2π	0	undefined

- In regions in which $\sin\,\theta$ increases, $\csc\,\theta$ will decrease.
- In regions in which $\sin\,\theta$ decreases, $\csc\,\theta$ will increase.
- When $\sin\,\theta = 1$, the reciprocal $\csc\,\theta = 1$.
- When $\sin\,\theta = -1$, the reciprocal $\csc\,\theta = -1$.
- When $\sin\,\theta = 0$, the reciprocal $\csc\,\theta$ is undefined.

Notice that $y = \csc\,\theta$ is undefined at 0, π, and 2π, and that no point exists. As θ approaches these values, the value of $\csc\,\theta$ becomes an extremely large positive value or an extremely small negative value.



For $y = \csc\theta$,

Domain: $\{\theta | \theta \neq n\pi,\ n \in I\}$ is read as follows:

The set of all θ such that θ cannot equal n times π radians, where n is a member of the integers.

Range: $\{y | y \leq -1 \text{ or } y \geq 1\}$ is read as follows:

The set of all values of y such that y is less than or equal to -1 or y is greater than or equal to 1.

$\csc\pi = \frac{1}{\sin\pi} = \frac{1}{0}$ and similarly, $\csc 2\pi = \frac{1}{0}$.

Since $\frac{1}{0}$ is not defined, you can conclude that $\csc\pi$ does not exist. The lines $\theta = \pi$, 2π, and so on are vertical asymptotes. The graphs for $\csc\theta$ extend indefinitely up and down, approaching these lines, but never touching them.

Therefore, when θ is a multiple of π rad, the function $y = \csc\theta$ is undefined. Hence, the domain of $y = \csc\theta$ is $\{\theta | \theta \neq n\pi,\ n \in I\}$. Recall that the range of $y = \sin\theta$ is $\{y | -1 \leq y \leq 1\}$. For $y = \csc\theta$, the range is $\{y | y \leq -1 \text{ or } y \geq 1\}$. The period of $y = \csc\theta$ is 2π rad.

Use a graphing calculator to draw $y = \frac{1}{\sin\theta}$. How does your graph compare with the preceding sketch?

1. a. Complete the following chart.

θ	$\cos\theta$	$\sec\theta$
0	1	
$0 < \theta < \frac{\pi}{2}$	positive and decreasing	
$\frac{\pi}{2}$	0	
$\frac{\pi}{2} < \theta < \pi$	negative and decreasing	
π	-1	
$\pi < \theta < \frac{3\pi}{2}$	negative and increasing	
$\frac{3\pi}{2}$	0	
$\frac{3\pi}{2} < \theta < 2\pi$	positive and increasing	
2π	1	

b. Using the same axis, sketch the graphs $y = \cos\ \theta$ and $y = \sec\ \theta$.

2. For $y = \sec\ \theta$, state the domain, the range, and the period.

3. By either utilizing the reciprocal relationship $\cot\ \theta = \frac{1}{\tan\ \theta}$ or by setting up a table of values, sketch the graph of $y = \cot\ \theta$, where $0 \le \theta \le 2\pi$.

4. For $y = \cot\ \theta$, state the domain, the range, and the period.

Check your answers by turning to the Appendix.

Here is a summary of the cosecant, secant, and cotangent graphs.

csc θ

- The graph lies within vertical asymptotes.
- The domain is $\{\theta | \theta \ne n\pi,\ n \in I\}$.
- The range is $\{y | y \le -1 \text{ or } y \ge 1\}$.
- The period is 2π rad.

sec θ

- The graph lies within vertical asymptotes.
- The domain is $\left\{\theta \middle| \theta \ne \frac{\pi}{2} + n\pi,\ n \in I\right\}$.
- The range is $\{y | y \le -1 \text{ or } y \ge 1\}$.
- The period is 2π rad.

cot θ

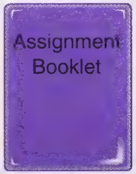
- The graph lies within vertical asymptotes.
- The domain is $\{\theta | \theta \ne n\pi,\ n \in I\}$.
- The range is R.
- The period is π rad.

Conclusion

In this section you graphed the six trigonometric functions. You described those graphs using the concepts of domain, range, and period. In addition, for the sine and cosine functions, you investigated amplitude, phase shift, and vertical translation. For the functions $y = a \sin\left[b(\theta + c)\right] + d$ and $y = a \cos\left[b(\theta + c)\right] + d$, you discovered that changes to the parameters a, b, c, and d affect the shape, size, and position of their corresponding graphs. You should be able to describe those changes in terms of those parameters.

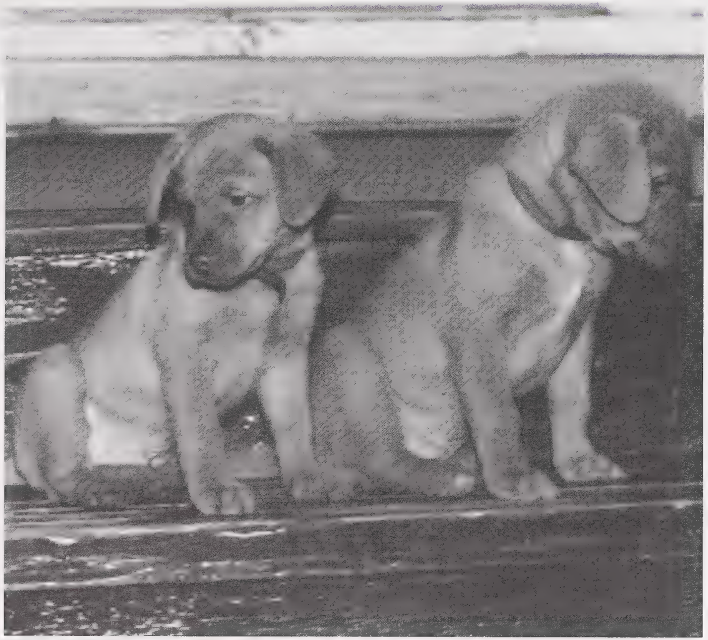
Periodic phenomena such as waves crashing on a beach, ocean tides, swinging a pendulum ball, or riding on a Ferris wheel can be modelled using sines or cosines. The next time you are at the beach, think, as you relax near the water, that the answer to the "good life" lies in trigonometry.

Assignment



You are now ready to complete the section assignment.

Section 5: Trigonometric Identities



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The occurrence of twins in a human family is uncommon. In some animals, it is common to have a pair of offspring that look alike. Are there twins in your family? Do you know of identical twins?

The algebraic equation $x + 3 = 8$, where $x \in R$ is true for only one value of the variable ($x = 5$). An equation like $2x = x + x$ is true for all values of the variable. It is not always obvious that both sides of such an equation are equal, so a proof is required. Since a proof involves showing that the left side is identical to the right side, these sentences are called identities. Identities are common in trigonometry, and proving these identities is a form of problem solving.

In this section you will prove some of the fundamental identities; then you will use these basic identities to prove more complex identities. You will study reciprocal identities, quotient identities, and Pythagorean identities. These fundamental trigonometric identities are used to simplify, evaluate, and derive alternate forms for trigonometric expressions. Finally, you will develop the addition and subtraction formulas of sine and cosine, and then use them to simplify trigonometric expressions.

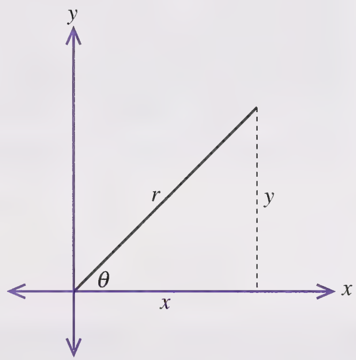
Activity 1: Quotient Identities, Reciprocal Identities, and Pythagorean Identities

Recipes may be given in metric or imperial measurements. A cook may change the measurements given from the metric form (millilitres, litres, and so on) to an equivalent measurement in the imperial form (teaspoons, cups, and so on), or vice versa. The cook may find it easier to work with a familiar measurement.

Just as in following a recipe, it is often helpful to change trigonometric expressions from one form to an equivalent form that is easier to work with. An equation that equates the two equivalent trigonometric expressions is called a **trigonometry identity**.

The definitions of the six trigonometric functions involve only three quantities: x, y, and r. The formula $x^2 + y^2 = r^2$ shows that the trigonometric functions are related.

Recall this diagram.



By definition, the following is true.

$\sin\,\theta = \frac{y}{r}$ $\qquad$ $\cos\,\theta = \frac{x}{r}$

$\tan\,\theta = \frac{y}{x}$, where $x \neq 0$ $\qquad$ $\csc\,\theta = \frac{r}{y}$, where $y \neq 0$

$\sec\,\theta = \frac{r}{x}$, where $x \neq 0$ $\qquad$ $\cot\,\theta = \frac{x}{y}$, where $y \neq 0$

There are three sets of basic identities: quotient identities, reciprocal identities, and Pythagorean identities. These basic identities are used to prove other, more complex identities. Look at the first set of these identities.

Quotient Identities

The following examples prove the quotient identities.

Example 1

Prove that for all θ, $\tan\,\theta = \frac{\sin\,\theta}{\cos\,\theta}$.

Solution

For all θ, $\sin\,\theta = \frac{y}{r}$, $\cos\,\theta = \frac{x}{r}$, and $\tan\,\theta = \frac{y}{x}$.

LS	RS
$\tan\theta$	$\dfrac{\sin\theta}{\cos\theta}$
$=\dfrac{y}{x}$	$=\dfrac{\frac{y}{r}}{\frac{x}{r}}$
	$=\dfrac{y}{x}$

LS = RS

This identity has one peculiarity that should be observed carefully. $\tan\theta$ is not defined for $x=0$ (which occurs when θ is coterminal with $\frac{\pi}{2}$ or $\frac{3\pi}{2}$).

The expression $\frac{\sin\theta}{\cos\theta}$ is not defined for the angles $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ because $\cos\frac{\pi}{2}=\cos\frac{3\pi}{2}=0$. Since the two sides of the equation $\tan\theta=\frac{\sin\theta}{\cos\theta}$ are equal for every value of θ for which the two sides are defined, the equation is called an **identity**.

Example 2

Prove that for all θ, $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$.

Solution

For all θ, $\sin\theta=\frac{y}{r}$, $\cos\theta=\frac{x}{r}$, and $\cot\theta=\frac{x}{y}$.

LS	RS
$\cot\theta$	$\dfrac{\cos\theta}{\sin\theta}$
$=\dfrac{x}{y}$	$=\dfrac{\frac{x}{r}}{\frac{y}{r}}$
	$=\dfrac{x}{y}$

LS = RS

The quotient identities are $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$.

Use a calculator to answer question 1.

1. Verify the following with $\theta=120°$.

   a. $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$

   b. $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$

Check your answers by turning to the Appendix.

Reciprocal Identities

Corresponding to every real number a $(a \neq 0)$, there exists a real number b $(b \neq 0)$ such that $a \times b = 1$. Each is called the reciprocal (or multiplicative inverse) of the other.

If $a \times b = 1$, then $a = \frac{1}{b}$ and $b = \frac{1}{a}$.

You will prove two of these identities in the following examples.

Example

Show that $\sin\theta = \dfrac{1}{\csc\theta}$ and $\csc\theta = \dfrac{1}{\sin\theta}$.

Solution

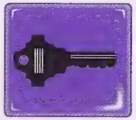
$$\csc\theta = \frac{r}{y}$$
$$\frac{1}{\csc\theta} = \frac{1}{\frac{r}{y}}$$
$$\frac{1}{\csc\theta} = \frac{y}{r}$$
$$\frac{1}{\csc\theta} = \sin\theta$$

$$\sin\theta = \frac{y}{r}$$
$$\frac{1}{\sin\theta} = \frac{1}{\frac{y}{r}}$$
$$\frac{1}{\sin\theta} = \frac{r}{y}$$
$$\frac{1}{\sin\theta} = \csc\theta$$

Similarly, you can show that

- $\cos\theta = \dfrac{1}{\sec\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$
- $\tan\theta = \dfrac{1}{\cot\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}$

Once these reciprocal identities are proved, they may be used to prove other identities.



The reciprocal identities are as follows:

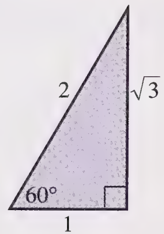
- $\sin\theta = \dfrac{1}{\csc\theta}$ or $\csc\theta = \dfrac{1}{\sin\theta}$
- $\cos\theta = \dfrac{1}{\sec\theta}$ or $\sec\theta = \dfrac{1}{\cos\theta}$
- $\tan\theta = \dfrac{1}{\cot\theta}$ or $\cot\theta = \dfrac{1}{\tan\theta}$

2. From the given right triangle with $\theta = 60°$, use exact values to verify the following reciprocal identities.

a. $\sin\theta = \dfrac{1}{\csc\theta}$

b. $\sec\theta = \dfrac{1}{\cos\theta}$

c. $\tan\theta = \dfrac{1}{\cot\theta}$



3. Write an equivalent expression in simplest form.

a. $\tan\theta \csc\theta$ **b.** $\dfrac{\tan\theta}{\sec\theta}$

4. Prove the following:

$$\sec\theta \csc\theta \cot\theta = \frac{1}{\sin^2\theta}$$

Check your answers by turning to the Appendix.

The reciprocal and quotient identities are rather like theorems in geometry because they are used to prove other identities. The Pythagorean identities are derived from the basic definition of the six trigonometric functions in terms of the arms and hypotenuse of a right triangle on the coordinate system.

Pythagorean Identities

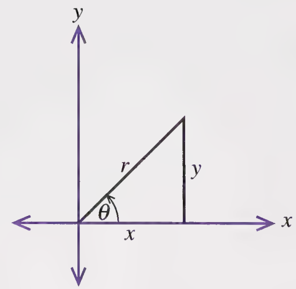
In the example that follows, the most commonly used identity will be proven.

Example

Show that $\sin^2\theta + \cos^2\theta = 1$.

Solution

From the diagram, you can write the identity.



$$y^2 + x^2 = r^2 \qquad \text{(Pythagorean Theorem)}$$

$$\frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{r^2}{r^2} \qquad \text{(Divide by } r^2\text{.)}$$

$$\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = 1$$

Since $\frac{y}{r} = \sin\theta$ and $\frac{x}{r} = \cos\theta$, then $\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \sin^2\theta + \cos^2\theta$.

Therefore, $\sin^2\theta + \cos^2\theta = 1$.

The other two Pythagorean identities are $\tan^2\theta + 1 = \sec^2\theta$ and $\cot^2\theta + 1 = \csc^2\theta$.

You should be able to use the Pythagorean identities in any of these forms:

- $\sin^2\theta + \cos^2\theta = 1$,
  $\sin^2\theta = 1 - \cos^2\theta$, or
  $\cos^2\theta = 1 - \sin^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$,
  $1 = \sec^2\theta - \tan^2\theta$, or
  $\tan^2\theta = \sec^2\theta - 1$
- $1 + \cot^2\theta = \csc^2\theta$,
  $1 = \csc^2\theta - \cot^2\theta$, or
  $\cot^2\theta = \csc^2\theta - 1$

Use a calculator to answer question 5.

5. Verify the following with $\theta = 120°$.

a. $\sin^2\theta + \cos^2\theta = 1$

b. $\sec^2\theta - \tan^2\theta = 1$

c. $\tan\theta \ \cot\theta = 1$

6. State whether the following are **True** or **False**.

a. $\cos\theta + \sin\theta = 1$

b. $\csc^2\theta - 1 = \cot^2\theta$

c. $\cot\theta = \dfrac{1}{\tan\theta}$

7. Prove the identity $\tan^2\theta + 1 = \sec^2\theta$.

Check your answers by turning to the Appendix.

Using the Three Sets of Basic Identities

Now you will prove other identities using the eight fundamental identities.

- $\sin\theta = \dfrac{1}{\csc\theta}$
- $\tan\theta = \dfrac{1}{\cot\theta}$
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
- $1 + \tan^2\theta = \sec^2\theta$
- $\cos\theta = \dfrac{1}{\sec\theta}$
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \cot^2\theta = \csc^2\theta$

The fundamental identities can be rearranged to determine other identities. Study the next example.

Example 1

Rearrange the identity $\cos\theta = \frac{1}{\sec\theta}$ to obtain $\sec\theta$.

Solution

$$\cos\theta = \frac{1}{\sec\theta}$$
$$\cos\theta\ \sec\theta = 1$$
$$\sec\theta = \frac{1}{\cos\theta}$$

Example 2

Rearrange the identity $\sin^2\theta + \cos^2\theta = 1$ to obtain $\sin^2\theta$.

Solution

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin^2\theta = 1 - \cos^2\theta$$

Simplifying trigonometric expressions will usually involve making substitutions from the eight identities. Rearrangement of terms may be necessary. Carefully study the next example.

Example 3

Use the eight fundamental identities to express the following in terms of a single trigonometric function.

- $\sin\theta\ \csc\theta\ \cos\theta$
- $\dfrac{\sin^2\theta}{\cos^2\theta} + 1$
- $\tan\alpha\ \cos\alpha\ \cot\alpha$
- $\cos^2\theta + \csc^2\theta + \sin^2\theta - 1$

Solution

$$\begin{aligned}\sin\theta\ \csc\theta\ \cos\theta &= \cancel{\sin\theta}\left(\frac{1}{\cancel{\sin\theta}}\right)\cos\theta \\ &= \cos\theta\end{aligned}$$

(Substitute $\csc\theta = \frac{1}{\sin\theta}$.)

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1$$

$\sin^2\theta = (\sin\theta)^2$

$$= \tan^2\theta + 1 \qquad \text{(quotient identity)}$$
$$= \sec^2\theta \qquad \text{(Pythagorean identity)}$$

$$\begin{aligned}\tan\alpha\ \cos\alpha\ \cot\alpha &= \left(\frac{\cancel{\sin\alpha}}{\cancel{\cos\alpha}}\right)(\cos\alpha)\left(\frac{\cancel{\cos\alpha}}{\cancel{\sin\alpha}}\right) \qquad \text{(quotient identity)} \\ &= \cos\alpha\end{aligned}$$

$$\cos^2\theta + \csc^2\theta + \sin^2\theta - 1 = \left(\cos^2\theta + \sin^2\theta\right) + \csc^2\theta - 1$$

$$= 1 + \csc^2\theta - 1 \quad \text{(Pythagorean identities)}$$

$$= \csc^2\theta$$

or

$$\cos^2\theta + \csc^2\theta + \sin^2\theta - 1 = \left(\cos^2\theta + \sin^2\theta\right) + \left(\csc^2\theta - 1\right)$$

$$= 1 + \cot^2\theta \quad \text{(Pythagorean identities)}$$

$$= \csc^2\theta$$

8. Write each of the following in terms of $\cos\theta$.

a. $\sin^2\theta$ **b.** $\sec^2\theta$

c. $\tan\theta\ \sin\theta$ **d.** $\dfrac{\csc\theta}{\cot\theta}$

9. Express each of the following in terms of $\sin\theta$, $\cos\theta$, or both.

a. $\dfrac{\tan\theta}{\sec\theta}$ **b.** $\sec\theta - \cos\theta$

c. $\dfrac{1+\cot^2\theta}{\sec^2\theta}$ **d.** $\dfrac{\csc\theta}{\tan\theta}$

e. $\cot\theta + \sec\theta$ **f.** $\dfrac{1}{1+\cot^2\theta}$

10. Express as a monomial.

a. $\tan^2\theta - \sec^2\theta$ **b.** $\sin y\left(1+\cot^2 y\right)$

c. $\cos x + \sin x\ \tan x$ **d.** $\dfrac{\sec D}{\tan D}$

Check your answers by turning to the Appendix.

There may be more than one method that will work. Try to find the **shortest** and **easiest** method. The previous examples involved relatively simple substitution and algebraic operations to simplify. Look at some more complex examples. Remember, the objective is to try to use the fewest number of steps possible and to keep the expression as simple as possible.

Three ways in which an identity may be verified are as follows:

- by changing the left member directly into the form of the right member
- by changing the right member directly into the form of the left member
- by changing both the left and right members into the same form, and doing each separately and independently of the other

Before proving an identity, remember the following:

- Study the whole identity carefully and try to discover whether any direct applications of the fundamental relations would be helpful. (For example, when squares of trigonometric functions are involved, try the Pythagorean identities.)

- If you are unable to determine the particular fundamental identity that should be applied, change all trigonometric functions to sines and/or cosines. (For example, replace $\tan\theta$ by $\frac{\sin\theta}{\cos\theta}$, $\sec\theta$ by $\frac{1}{\cos\theta}$, and so on.)

- It is advisable to begin with the side (or member) of the identity that has the greater number of terms and is, therefore, more complicated.

- An identity is an open sentence whose solution set is the same as the domain of the variable except for values of the variable for which either side of the equation is not defined.

The following are examples that prove identities that are a little more complex.

Example 4

Prove the following identities.

- $\csc\theta - \cos\theta\ \cot\theta = \sin\theta$
- $\cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$
- $\dfrac{1}{1+\cos\theta} + \dfrac{1}{1-\cos\theta} = 2\csc^2\theta$
- $\tan A(\tan A + \cot A) = \sec^2 A$
- $\dfrac{1+\tan\theta}{1+\cot\theta} = \dfrac{1-\tan\theta}{\cot\theta - 1}$
- $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\ \cos\theta$

Solution

$$\begin{aligned}\csc\theta - \cos\theta\ \cot\theta &= \frac{1}{\sin\theta} - \cos\theta\left(\frac{\cos\theta}{\sin\theta}\right)\\ &= \frac{1-\cos^2\theta}{\sin\theta}\\ &= \frac{\sin^2\theta}{\sin\theta}\\ &= \sin\theta\end{aligned}$$

$$\therefore\ \csc\theta - \cos\theta\ \cot\theta = \sin\theta$$

$$\begin{aligned}\cos^2\theta - \sin^2\theta &= \cos^2\theta - \left(1-\cos^2\theta\right)\\ &= \cos^2\theta - 1 + \cos^2\theta\\ &= 2\cos^2\theta - 1\end{aligned}$$

$$\therefore\ \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$

$$\begin{aligned}
&\frac{1}{1+\cos\theta}+\frac{1}{1-\cos\theta}\\
&=\left(\frac{1}{1+\cos\theta}\right)\left(\frac{1-\cos\theta}{1-\cos\theta}\right)+\left(\frac{1}{1-\cos\theta}\right)\left(\frac{1+\cos\theta}{1+\cos\theta}\right)\\
&=\frac{(1-\cos\theta)+(1+\cos\theta)}{(1+\cos\theta)(1-\cos\theta)}\\
&=\frac{2}{(1+\cos\theta)(1-\cos\theta)}\\
&=\frac{2}{1-\cos^2\theta}\\
&=\frac{2}{\sin^2\theta}\\
&=2\csc^2\theta \qquad \left(\frac{1}{\sin\theta}=\csc\theta\right)
\end{aligned}$$

$$\therefore\ \frac{1}{1+\cos\theta}+\frac{1}{1-\cos\theta}=2\csc^2\theta$$

$$\begin{aligned}
\tan A(\tan A+\cot A)&=\tan^2 A+\tan A\cot A\\
&=\tan^2 A+\tan A\left(\frac{1}{\tan A}\right)\\
&=\tan^2 A+1\\
&=\sec^2 A
\end{aligned}$$

Thus, $\tan A(\tan A+\cot A)=\sec^2 A$.

Since both sides of $\frac{1+\tan\theta}{1+\cot\theta}=\frac{1-\tan\theta}{\cot\theta-1}$ are equally complicated, transform both sides to equivalent expressions.

LS	RS
$\dfrac{1+\tan\theta}{1+\cot\theta}$	$\dfrac{1-\tan\theta}{\cot\theta-1}$
$=\dfrac{1+\frac{\sin\theta}{\cos\theta}}{1+\frac{\cos\theta}{\sin\theta}}$	$=\dfrac{1-\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\sin\theta}-1}$
$=\dfrac{\frac{\cos\theta+\sin\theta}{\cos\theta}}{\frac{\sin\theta+\cos\theta}{\sin\theta}}$	$=\dfrac{\frac{\cos\theta-\sin\theta}{\cos\theta}}{\frac{\cos\theta-\sin\theta}{\sin\theta}}$
$=\dfrac{\cos\theta+\sin\theta}{\cos\theta}\times\dfrac{\sin\theta}{\sin\theta+\cos\theta}$	$=\dfrac{\cos\theta-\sin\theta}{\cos\theta}\times\dfrac{\sin\theta}{\cos\theta-\sin\theta}$
$=\dfrac{\sin\theta}{\cos\theta}$	$=\dfrac{\sin\theta}{\cos\theta}$

LS = RS

$$\therefore\ \frac{1+\tan\theta}{1+\cot\theta}=\frac{1-\tan\theta}{\cot\theta-1}$$

$$\begin{aligned}
&(\sin\theta+\cos\theta)^2\\
&=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta \qquad \left((a+b)^2=a^2+2ab+b^2\right)\\
&=\left(\sin^2\theta+\cos^2\theta\right)+2\sin\theta\cos\theta \qquad \text{(Group.)}\\
&=1+2\sin\theta\cos\theta \qquad \text{(Substitute.)}
\end{aligned}$$

$$\therefore\ (\sin\theta+\cos\theta)^2=1+2\sin\theta\cos\theta$$

You have studied numerous examples on various types of identities. This should give you a good idea of how to prove the identities in the questions that follow.

11. Show why the left side is identical to the right side for the following expressions.

a. $\dfrac{1-\sin^2\theta}{\csc^2\theta-1}=\sin^2\theta$

b. $\sec^2\theta-\sin^2\theta=\cos^2\theta+\tan^2\theta$

c. $\sec\theta\csc\theta\cot\theta=\csc^2\theta$

12. To prove some identities, you need to use your factoring skills. Factor each of the following.

a. $1-\cos^2\theta$

b. $\sin\alpha-\sin^2\alpha$

c. $\tan^4\beta-\cot^4\beta$

d. $4\cos^2 y-4\cos y+1$

e. $\sec x\sin x+\sec x$

f. $\sin\theta\sec\theta-\sin\theta+\sec\theta-1$

13. Use the necessary steps to prove the following identities.

a. $\cos A\tan A=\sin A$

b. $\sin^3 A+\sin A\cos^2 A=\dfrac{1}{\csc A}$

c. $\sec y-\tan y\sin y=\cos y$

d. $\cos^4\theta-\sin^4\theta=1-2\sin^2\theta$

e. $\dfrac{1}{1-\sin x}=\sec^2 x+\sec x\tan x$

f. $(1+\cot A)^2=\csc^2 A+2\cot A$

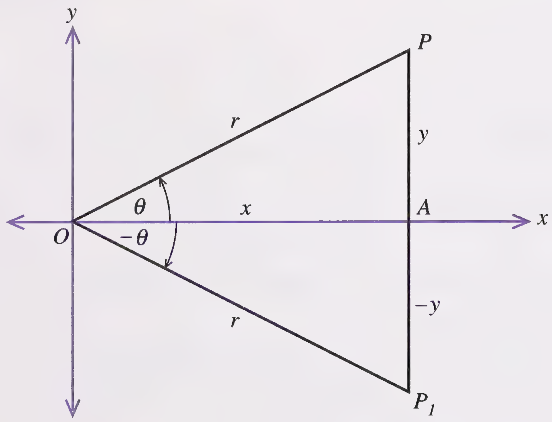
g. $\dfrac{\tan\theta-\sin\theta}{\tan\theta\sin\theta}=\dfrac{1-\cos\theta}{\sin\theta}$

Check your answers by turning to the Appendix.

In this activity you found that identities are statements of equality that are true for all values of the variable.

Activity 2: Negative and Complementary Angles

Consider the functions of θ, where θ is an angle of any magnitude. In the following diagram, θ and $-\theta$ are in standard position and P and P_1 are points on the terminal sides of the angles respectively, having the same distance r from the origin. The right triangles AOP and AOP_1 are congruent.



From the diagram,

$$\sin(-\theta) = \frac{-y}{r} = -\sin\theta \qquad \cos(-\theta) = \frac{x}{r} = \cos\theta \qquad \tan(-\theta) = \frac{-y}{x} = -\tan\theta$$

$$\csc(-\theta) = \frac{r}{-y} = -\csc\theta \qquad \sec(-\theta) = \frac{r}{x} = \sec\theta \qquad \cot(-\theta) = \frac{x}{-y} = -\cot\theta$$

Notice that $\cos(-\theta)$ and $\sec(-\theta)$ are the only two functions that become positive.

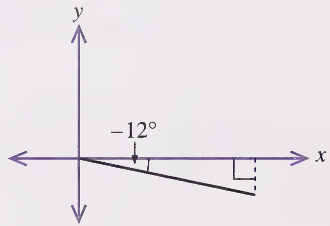
Study the next example.

Example 1

Express $\sin(-12°)$ in terms of the same function of a positive acute angle.

Solution

$\sin(-12°) = -\sin 12°$



Example 2

Express $\sec(-225°)$ in terms of the same function of a positive acute angle.

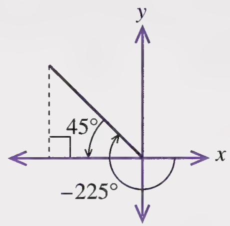
Solution

$\sec(-\theta) = \sec \theta$

$\therefore \sec(-225°) = \sec 225°$

$= \sec(180° + 45°)$

$= -\sec 45°$



In the second quadrant, $\sec \theta$ is negative.

1. Express each of the following in terms of the same function of a positive acute angle.

   a. $\tan(-15°)$ b. $\csc(-68°)$

   c. $\cos(-225°)$ d. $\cot(-166°)$

2. Calculate each of the following using exact values.

   a. $\sin(-30°)$ b. $\cot 135°$

   c. $\cos(-60°)$ d. $\csc(-120°)$

Check your answers by turning to the Appendix.

By using a calculator, it can be seen that the cosine of an angle equals the sine of the complement of the same angle. An example would be $\cos 35° = \sin 55°$. By using the definitions of the trigonometric functions, this can be illustrated.

In Figure 1, complementary acute angles θ and $90° - \theta$ are drawn in standard position, and perpendiculars are constructed from P and Q to the x-axis. The two triangles formed are congruent.

Standard position has one vertex at the origin.

All three corresponding sides are congruent (S.S.S.).

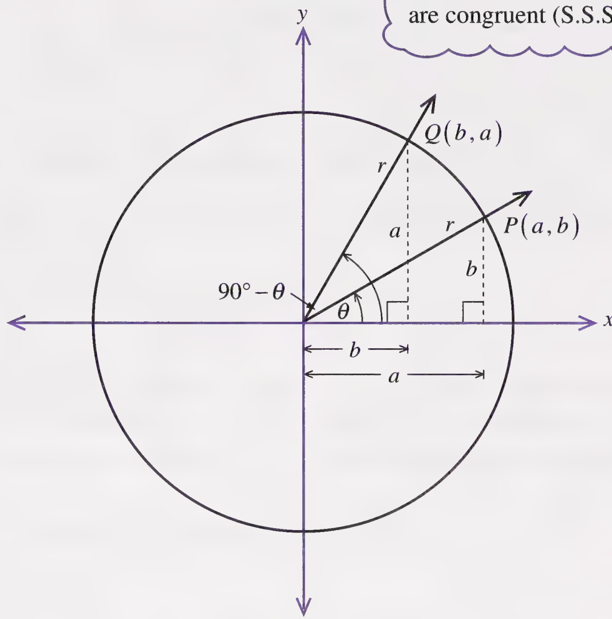


Figure 1

The two triangles are congruent since they both have right angles and the lengths of their sides are a, b, and r. Thus, the angles for each triangle are $90°$, θ, and $90° - \theta$.

$$\begin{aligned}\text{From the diagram, } \sin\theta &= \frac{y\text{-coordinate of } P}{\text{radius}} \\ &= \frac{b}{r} \\ &= \frac{x\text{-coordinate of } Q}{\text{radius}} \\ &= \cos(90° - \theta)\end{aligned}$$

Look at the unit circle again.

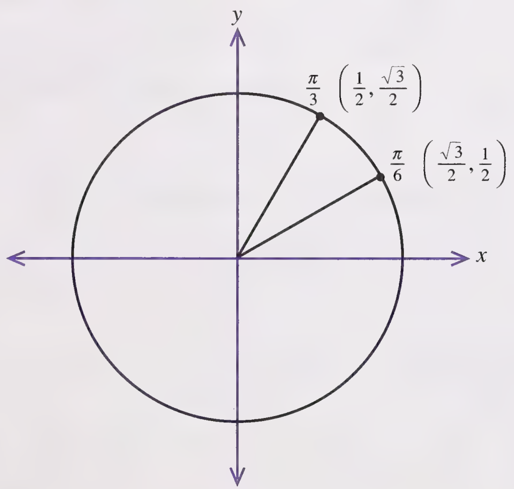


Figure 2

You will notice that $30°$ and $60°$ are complementary angles.

From Figure 2, $\cos\frac{\pi}{6} = \sin\frac{\pi}{3}$ since $\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$. Also, $\sin\frac{\pi}{6} = \cos\frac{\pi}{3}$ since $\frac{1}{2} = \frac{1}{2}$.

Other ratios for cofunctions are easily tested on the calculator.

Because $\sin\theta = \cos(90° - \theta)$, the cosine is called the cofunction of the sine. Restated, the sine of an angle of θ is the cofunction (cosine) of the complement $(90° - \theta)$ of θ. Likewise, the sine is the cofunction of the cosine.

From Figure 1, $\cos\theta = \frac{a}{r} = \sin(90° - \theta)$.

Figure 1 can also be used to prove the remaining five cofunctions.

The **cofunction relationships** are summarized as follows:

- $\sin\theta = \cos(90° - \theta)$ and $\cos\theta = \sin(90° - \theta)$
- $\tan\theta = \cot(90° - \theta)$ and $\cot\theta = \tan(90° - \theta)$
- $\sec\theta = \csc(90° - \theta)$ and $\csc\theta = \sec(90° - \theta)$

Any trigonometric function of a positive acute angle is equal to the cofunction of the complementary angle. The following are some examples which illustrate this:

- $\sin 60° = \cos 30°$
- $\tan 20° = \cot 70°$
- $\csc 45° = \sec 45°$
- $\sec(90° - 20°) = \csc 20°$

Now look at the following examples.

Example 3

Express $\cos 39°$ in terms of its cofunction.

Solution

The cofunction property for cosine is $\sin(90° - \theta) = \cos\theta$.

$$\begin{aligned} \cos 39° &= \sin(90° - 39°) \\ &= \sin 51° \end{aligned}$$

Example 4

Express $\tan\frac{\pi}{5}$ in terms of its cofunction.

Solution

$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$ (cofunction property)

$$\begin{aligned} \therefore\ \tan\frac{\pi}{5} &= \cot\left(\frac{\pi}{2} - \frac{\pi}{5}\right) \\ &= \cot\frac{3\pi}{10} \end{aligned}$$

Example 5

If $\theta > 0$, write $-\tan(-2-\theta)$ with a positive angle.

Solution

Using the negative angle properties, $\tan(-\theta) = -\tan\theta$.

$$\begin{aligned} -\tan(-2-\theta) &= -\tan[-(2+\theta)] \\ &= -[-\tan(2+\theta)] \\ &= \tan(2+\theta) \end{aligned}$$

3. Find $\cos A$ when $\sin(90° - A) = \frac{2}{5}$.

4. Find $\sec A$ when $\sin(90° - A) = \frac{2}{5}$.

5. If $\sin A = \cos 3A$, find A.
[Hint: $\sin A = \sin(90° - 3A)$]

6. If $\theta > 0$, write each with a positive angle.

a. $-\sin(-2\theta)$ **b.** $-\cot(-3-4\theta)$

7. Simplify the following by writing each in terms of a single trigonometric function with a single variable.

a. $\cos\,(90° - 5r)$ **b.** $\csc\left(\frac{\pi}{2} - 44k\right)$

c. $\sin\,(-90° + 5m)$ **d.** $\tan\left(-\frac{\pi}{2} + 7b\right)$

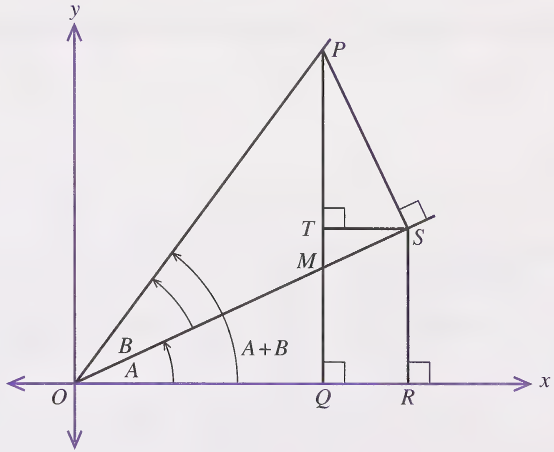
Check your answers by turning to the Appendix.

Trigonometric expressions can be simplified using negative angle relations and complementary angle relations.

Activity 3: The Sum and Difference Formulas

Other properties exist for functions which are the sum or difference of two angles.

Using the following diagram, the formula
$\cos\,(A+B) = \cos\,A\,\cos\,B - \sin\,A\,\sin\,B$ will be proven.



To derive formulas for functions of $A+B$, angles A and B are placed with reference to the coordinate axes. Taking P on the terminal arm of $(A+B)$, the following perpendiculars are drawn: $PS \perp OS$, PQ and SR perpendicular to the x-axis, and $ST \perp PQ$.

In ΔPOQ, $\frac{OQ}{OP} = \cos\,(A+B)$.

$\therefore\ OQ = OP\cos\,(A+B)$ 1

In ΔSOR, $\frac{OR}{OS} = \cos\,A$.

$\therefore\ OR = OS\cos\,A$ 2

ΔSPM is similar to ΔQOM because both triangles have right angles and their vertically opposite angles at M are equal. Thus, $\angle MPS = \angle A$.

In ΔPTS, $\frac{TS}{PS} = \sin A$.

$\therefore\ TS = PS \sin A$ (3)

$TSRQ$ is a rectangle.

But, $QR = TS$; therefore, $QR = PS \sin A$.

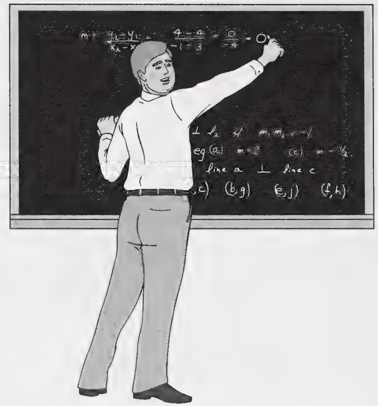
Substitute (1), (2), and (3) into $OQ = OR - QR$.

$$\therefore\ OP \cos(A+B) = OS \cos A - PS \sin A$$

$$\cos(A+B) = \frac{OS}{OP}\cos A - \frac{PS}{OP}\sin A \qquad \text{(Divide by } OP.)$$

$\frac{OS}{OP} = \cos B$ and $\frac{PS}{OP} = \sin B$

$$\therefore\ \cos(A+B) = \cos A \cos B - \sin A \sin B$$



Example 1

Using the exact values, verify the formula $\cos(A+B) = \cos A \cos B - \sin A \sin B$ for the following:

- $A = 90°$ and $B = 30°$
- $A = 210°$ and $B = 60°$

Solution

Use a LS/RS table for $A = 90°$ and $B = 30°$ to work out each side of the formula separately.

LS	RS
$\cos(A+B)$	$\cos A \cos B - \sin A \sin B$
$= \cos(90° + 30°)$	$= \cos 90° \cos 30° - \sin 90° \sin 30°$
$= \cos 120°$	$= 0\left(\frac{\sqrt{3}}{2}\right) - 1\left(\frac{1}{2}\right)$ (These numbers come from the unit circle.)
$= -\cos 60°$ (related angle)	
$= -\frac{1}{2}$ (unit circle)	$= -\frac{1}{2}$

LS = RS

$\therefore \cos(A+B) = \cos A \cos B - \sin A \sin B$

Use a LS/RS table for $A = 210°$ and $B = 60°$ to work out each side of the formula separately.

LS	RS
$\cos(A+B)$	$\cos A \cos B - \sin A \sin B$
$= \cos(210° + 60°)$	$= \cos 210° \cos 60° - \sin 210° \sin 60°$
$= \cos(270°)$	$= -\cos 30° \cos 60° - (-\sin 30° \sin 60°)$
$= \cos 90°$	$= -\cos \frac{\pi}{6} \cos \frac{\pi}{3} - \left(-\sin \frac{\pi}{6} \sin \frac{\pi}{3}\right)$
$= 0$	$= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$
	$= -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}$
	$= 0$

LS = RS

$\therefore \cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos 90° \cos 30° - \sin 90° \sin 30°$
$= \cos \frac{\pi}{2} \cos \frac{\pi}{6} - \sin \frac{\pi}{2} \sin \frac{\pi}{6}$

$210° = 180° + 30°$

$270° = 180° + 90°$

To derive a formula for $\cos\ (A-B)$, the formula can be written as follows:

$$\cos\ (A-B)=\cos\left[A+(-B)\right]$$
$$=\cos A\ \cos\ (-B)-\sin A\ \sin\ (-B)$$

$$\cos\ (-B)=\cos B$$
$$\sin\ (-B)=-\sin B$$

Using the negative angle properties,
$\cos\ (A-B)=\cos A\ \cos B+\sin A\ \sin B$.

To determine the formula for $\sin\ (A+B)$, apply the cofunction properties.

$$\sin\ (A+B)=\cos\left[90°-(A+B)\right]$$
$$=\cos\left[90°-A-B\right]$$
$$=\cos\left[(90°-A)-B\right]$$
$$=\cos\ (90°-A)\ \cos B+\sin\ (90°-A)\ \sin B$$
$$=\sin A\ \cos B+\cos A\ \sin B$$

$\therefore\ \sin\ (A+B)=\sin A\ \cos B+\cos A\ \sin B$

Similarly, $\sin\ (A-B)=\sin A\ \cos B-\cos A\ \sin B$.

To find the formula for $\tan\ (A+B)$, you can write the expression in terms of sine and cosine.

$$\tan\ (A+B)=\frac{\sin\ (A+B)}{\cos\ (A+B)}$$
$$=\frac{\sin A\ \cos B+\cos A\ \sin B}{\cos A\ \cos B-\sin A\ \sin B}$$

A convenient way to rewrite this in terms of tangent is to divide the numerator and denominator by $\cos A\ \cos B$.

$$\frac{\frac{\sin A\ \cos B}{\cos A\ \cos B}+\frac{\cos A\ \sin B}{\cos A\ \cos B}}{\frac{\cos A\ \cos B}{\cos A\ \cos B}-\frac{\sin A\ \sin B}{\cos A\ \cos B}}=\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{1-\frac{\sin A\ \sin B}{\cos A\ \cos B}}$$
$$=\frac{\tan A+\tan B}{1-\tan A\ \tan B}$$

$$\therefore\ \tan\ (A+B)=\frac{\tan A+\tan B}{1-\tan A\ \tan B}$$

Similarly, $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A \tan B}$.

Here is a summary of the sum and difference formulas.

- $\cos(A+B)=\cos A \cos B-\sin A \sin B$
- $\cos(A-B)=\cos A \cos B+\sin A \sin B$
- $\sin(A+B)=\sin A \cos B+\cos A \sin B$
- $\sin(A-B)=\sin A \cos B-\cos A \sin B$
- $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A \tan B}$
- $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A \tan B}$

1. By substituting 30° for a and 120° for b, show that the trigonometric function $\cos(a+b)\neq\cos a+\cos b$ (not distributive).

2. Simplify the following using the sum and difference formulas.

a. $\cos(\pi-\theta)$ **b.** $\tan\left(\frac{\pi}{4}+\theta\right)$ **c.** $\sin\left(\theta-\frac{\pi}{2}\right)$

3. Verify the sum formula of sine and the difference formula of cosine with the given angles.

a. $A=210°$ and $B=120°$ **b.** $A=\frac{5\pi}{6}$ and $B=\frac{\pi}{3}$

4. Express each of the following as a trigonometric function with a single angle or variable.

a. $\cos 40° \sin 125°-\sin 40° \cos 125°$

b. $\cos\frac{\pi}{6} \cos\frac{2\pi}{3}+\sin\frac{\pi}{6} \sin\frac{2\pi}{3}$

c. $\sin 5d \cos 2d+\cos 5d \sin 2d$

Check your answers by turning to the Appendix.

You will be shown how the sum and difference formulas apply to the next few examples.

Example 2

Find the exact value of $\cos 75°$.

Solution

Split 75° into two conventient values so that you can use the special angles (30°, 45°, 60°, and so on) of the unit circle.

Substitute the two convenient values into the formula for $\cos(A+B)$.

$$
\begin{aligned}
\cos 75^\circ &= \cos(45^\circ + 30^\circ) \\
&= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\
&= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \\
&= \frac{\sqrt{3}-1}{2\sqrt{2}} \\
&= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}
$$

Check

Using a calculator, find $\cos 75^\circ$; then evaluate $\frac{\sqrt{3}-1}{2\sqrt{2}}$.

7 5 cos

0.2588 19045

3 √ − 1 = ÷ (

2 × 2 √) =

0.2588 19045

The answers are the same.

You could have found the exact value of $\cos 75^\circ$ using other combinations such as $\cos(135^\circ - 60^\circ)$.

Example 3

Use a trigonometric subtraction formula to find $\sin 15^\circ$. Express your answer in simplest radical form.

Solution

Let $15^\circ = 60^\circ - 45^\circ$.

Use the formula for $\sin(A-B)$.

$$\begin{aligned}\sin 15° &= \sin(60°-45°)\\ &= \sin 60° \cos 45° - \cos 60° \sin 45°\\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)-\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Therefore, $\sin 15° = \frac{\sqrt{6}-\sqrt{2}}{4}$.

5. Find the exact value of the following using the sum and difference formulas.

a. $\cos 105°$ **b.** $\cos \frac{\pi}{12}$

6. If $\cos\theta = \frac{3}{5}$ and $\cos\beta = \frac{12}{13}$, find the value of $\sin(\beta-\theta)$. θ and β are both acute angles and $\beta < \theta$.

7. Evaluate $\sin(-15°)$ using the trigonometric addition or subtraction formulas.

8. Find the exact values for each of the following using the trigonometric addition formulas.

a. $\sin 285°$ **b.** $\cos 255°$

Check your answers by turning to the Appendix.

The following example shows you how to simplify a trigonometric function containing a double angle.

Example 4

Find $\sin 2\theta$ if $\sin\theta = \frac{3}{5}$.

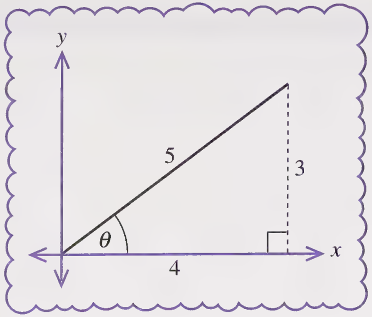
Solution

Using the formula for $\sin(A+B)$,

$$\begin{aligned}\sin 2\theta &= \sin(\theta+\theta)\\ &= \sin\theta\cos\theta+\cos\theta\sin\theta\\ &= 2\sin\theta\cos\theta\end{aligned}$$

To find the value of $\cos\theta$, use the Pythagorean identity.

$$\sin^2\theta + \cos^2\theta = 1$$
$$\left(\frac{3}{5}\right)^2 + \cos^2\theta = 1$$
$$\cos^2\theta = 1 - \frac{9}{25}$$
$$\cos^2\theta = \frac{16}{25}$$
$$\cos\theta = \frac{4}{5}$$



$$\therefore\ \sin(2\theta) = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}$$

9. Find $\cos 2x$ when $\sin x = \frac{4}{5}$ and $90° < x < 180°$.

Check your answers by turning to the Appendix.

The sum and difference formulas may be used in proving identities. The next example shows you this.

Example 5

Verify the identity $\tan B = \dfrac{1-\cos 2B}{\sin 2B}$.

Solution

Equate the right side since it is more complex, and use the sum formulas.

$$\begin{aligned}
\frac{1-\cos 2B}{\sin 2B} &= \frac{1-\cos(B+B)}{\sin(B+B)} \\
&= \frac{1-(\cos B\cos B - \sin B\sin B)}{\sin B\cos B + \cos B\sin B} \\
&= \frac{1-\left(\cos^2 B - \sin^2 B\right)}{2\sin B\cos B} \\
&= \frac{1-\cos^2 B + \sin^2 B}{2\sin B\cos B} \\
&= \frac{\sin^2 B + \sin^2 B}{2\sin B\cos B} \\
&= \frac{2\sin^2 B}{2\sin B\cos B} \\
&= \frac{\sin B}{\cos B} \\
&= \tan B
\end{aligned}$$

$$\therefore\ \tan B = \frac{1-\cos 2B}{\sin 2B}$$

10. Verify the given identities.

a. $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$ **b.** $\csc 2x = \cot x - \cot 2x$

11. State (using exact values) whether the following are **True** or **False**.

a. $\cos 120° = 2\cos^2 60° - 1$

b. $\sin(-120°) = 2\sin 60° \cos(-60°)$

c. $\sin(30° + 60°) = \sin 30° + \sin 60°$

d. $\tan(-150°) = \tan(150°)$

e. $\sec(30°) = \csc(60°)$

Check your answers by turning to the Appendix.

You can find the value of $\sin(\alpha \pm \beta)$ or $\cos(\alpha \pm \beta)$ if you are given the values of $\sin \alpha$ and $\sin \beta$.

The following example shows this.

Example 6

Given the acute angles α and β, where $\sin \alpha = \frac{5}{13}$ and $\sin \beta = \frac{4}{5}$, find $\sin(\alpha + \beta)$.

Solution

Since $\sin(\alpha+\beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$, you must determine $\cos \alpha$ and $\cos \beta$. Use the fact that $\sin^2 \theta + \cos^2 \theta = 1$ for any angle.

$$\sin \alpha = \frac{5}{13}$$

$$\sin^2 \alpha = \frac{25}{169}$$

$$\sin^2 \alpha + \cos^2 \alpha = 1$$

$$\cos^2 \alpha = 1 - \frac{25}{169}$$

$$\cos^2 \alpha = \frac{144}{169}$$

$$\cos \alpha = \sqrt{\frac{144}{169}}$$

$$= \pm\frac{12}{13}$$

Since α is acute, then $\cos \alpha = \frac{12}{13}$.

$$\sin\beta = \frac{4}{5}$$
$$\sin^2\beta = \frac{16}{25}$$
$$\sin^2\beta + \cos^2\beta = 1$$
$$\cos^2\beta = 1 - \frac{16}{25}$$
$$\cos^2\beta = \frac{9}{25}$$
$$\cos\beta = \sqrt{\frac{9}{25}}$$
$$= \pm\frac{3}{5}$$

Since β is acute, then $\cos\beta = \frac{3}{5}$.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(\frac{5}{13}\right)\left(\frac{3}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{4}{5}\right) \\ &= \frac{15}{65} + \frac{48}{65} \\ &= \frac{63}{65}\end{aligned}$$

12. α and β are acute angles with $\beta > \alpha$. Find $\sin(\alpha - \beta)$ when $\sin\alpha = \frac{5}{13}$ and $\sin\beta = \frac{4}{5}$.

13. Given the acute angles A and B, where $\sin A = \frac{12}{13}$ and $\cos B = \frac{3}{5}$, find $\cos(A - B)$.

Check your answers by turning to the Appendix.

A calculator is useful and quick in evaluating trigonometric expressions. The addition and subtraction identity formulas are helpful in simplifying trigonometric expressions or giving the solutions of trigonometric equations as exact values.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

You might have realized that trigonometric functions that have negative acute angles can be changed to positive acute angles and the signs of such functions are easily determined. When the negative angles are less than $-90°$, determining trigonometric functions with positive acute angles requires a little more work. It will help you a great deal if you memorize the trigonometric functions that are positive in each quadrant.

Study this example.

Example 1

Express $\cos(-140°)$ in terms of the same function of a positive acute angle.

Solution

$\cos(-140°)$ is in the third quadrant; therefore, it is negative.

$$\begin{aligned}\cos(-140°) &= \cos(180° - 140°)\\ &= -\cos 40°\end{aligned}$$

1. By inspection, state if each of the following functions is **positive** or **negative**.

   a. $\tan(-272°)$ b. $\csc(-96°)$
   c. $\cos(-285°)$ d. $\sec(-190°)$

2. Express each in terms of the same function of a positive acute angle using the method described in Example 1.

   a. $\cot(-100°)$ b. $\cos(-269°)$
   c. $\tan(-230°)$ d. $\sin(-150°)$

Check your answers by turning to the Appendix.

In Activity 2 of this section, you learned about cofunction relations. In the same way, you can simplify trigonometric functions that make use of 180°. In the next example, the sum or difference formula will be applied in order to simplify a trigonometric function.

Example 2

Simplify $\cos(180° - \theta)$.

θ is always assumed to be an acute angle.

Solution

Use the formula for $\cos(\alpha - \beta)$.

Using $\cos 180° = -1$ and $\sin 180° = 0$, complete the following statement.

$$\begin{aligned}\cos(180° - \theta) &= \cos 180°(\cos\theta) + \sin 180°(\sin\theta)\\ &= -\cos\theta + (0)\sin\theta\\ &= -\cos\theta + 0\\ &= -\cos\theta\end{aligned}$$

You can simplify $\cos(180° - \theta)$ without using the sum and difference formulas. Again, if you can remember the sign of each trigonometric function in its respective quadrant, you can conclude its simplified form. $\cos(180° - \theta)$ lies in the second quadrant. Therefore, the sign is negative and $\cos(180° - \theta) = -\cos\theta$. Similarly, $\sin(180° - \theta) = \sin\theta$.

Here is a summary of the six functions using 180°.

- $\sin(180° - \theta) = \sin\theta$
  $\sin(180° + \theta) = -\sin\theta$
- $\tan(180° - \theta) = -\tan\theta$
  $\tan(180° + \theta) = \tan\theta$
- $\sec(180° - \theta) = -\sec\theta$
  $\sec(180° + \theta) = -\sec\theta$
- $\cos(180° - \theta) = -\cos\theta$
  $\cos(180° + \theta) = -\cos\theta$
- $\cot(180° - \theta) = -\cot\theta$
  $\cot(180° + \theta) = \cot\theta$
- $\csc(180° - \theta) = \csc\theta$
  $\csc(180° + \theta) = -\csc\theta$

3. Use the formula for $\cos(\alpha + \beta)$ to prove $\cos(180° + \theta) = -\cos\theta$.

4. Use the formula for $\sin(A - B)$ to prove $\sin(180° - \theta) = \sin\theta$.

Check your answers by turning to the Appendix.

Enrichment

Some identities are more complex than others and proving them becomes a tedious task. In such cases, it may be necessary to prove each side partially until the results match.

The following example shows you one such complex identity.

Example

Prove $\dfrac{1+\sec\theta}{\sec\theta} = \dfrac{\sin^2\theta}{1-\cos\theta}$.

Solution

LS	RS
$\dfrac{1+\sec\theta}{\sec\theta}$	$\dfrac{\sin^2\theta}{1-\cos\theta}$
$=\dfrac{1+\frac{1}{\cos\theta}}{\frac{1}{\cos\theta}}$	$=\dfrac{1-\cos^2\theta}{1-\cos\theta}$
$=\dfrac{\frac{\cos\theta+1}{\cos\theta}}{\frac{1}{\cos\theta}}$	$=\dfrac{\cancel{(1-\cos\theta)}(1+\cos\theta)}{\cancel{(1-\cos\theta)}}$
$=\cos\theta+1$	$=1+\cos\theta$

LS = RS

1. Prove the following identities.

a. $\dfrac{\tan x}{\sec x - \tan x} = \sec^2 x + \sec x \tan x - 1$

b. $\sec x \cos x = \tan x \cot x$

c. $\dfrac{\tan^2\theta + 1}{\cot^2\theta + 1} = \tan^2\theta$

d. $\dfrac{\cos 2x}{1+\sin 2x} = \dfrac{\cot x - 1}{\cot x + 1}$

2. Simplify these expressions by writing each in terms of a single trigonometric function with a single variable.

   a. $\dfrac{\tan m - \tan 6m}{1 + \tan m \ \tan 6m}$

   b. $\cos 7x \ \cos 2x - \sin 7x \ \sin 2x$

3. Find the exact value of $\tan \frac{19\pi}{6}$ using the sum and/or difference formulas.

4. Prove that $\cos(x+y)\ \cos(x-y) = \cos^2 x + \cos^2 y - 1$.

5. Show that $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$.

6. Find an expression for $\cos 3x$.

7. Verify the sum property for sine when $A = \frac{4\pi}{3}$ and $B = -\frac{5\pi}{6}$.

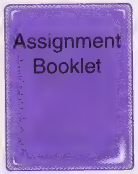
Check your answers by turning to the Appendix.

Conclusion

In this section you derived the quotient identities, the reciprocal identities, and the Pythagorean identities. These identities were used to expand or simplify more complex trigonometric expressions. You then derived the negative-angle and complimentary-angle identities. Also, you investigated the sum and difference formulas.

Remember, an identity is a statement in which the left-hand and right-hand sides are "identical" for all permissible replacements of the variable. The two sides of an identity are like identical twins, each wearing a different disguise. To expose them, to show that they are the "same," you must remove those disguises by applying the principals of this section and the rules of algebra.

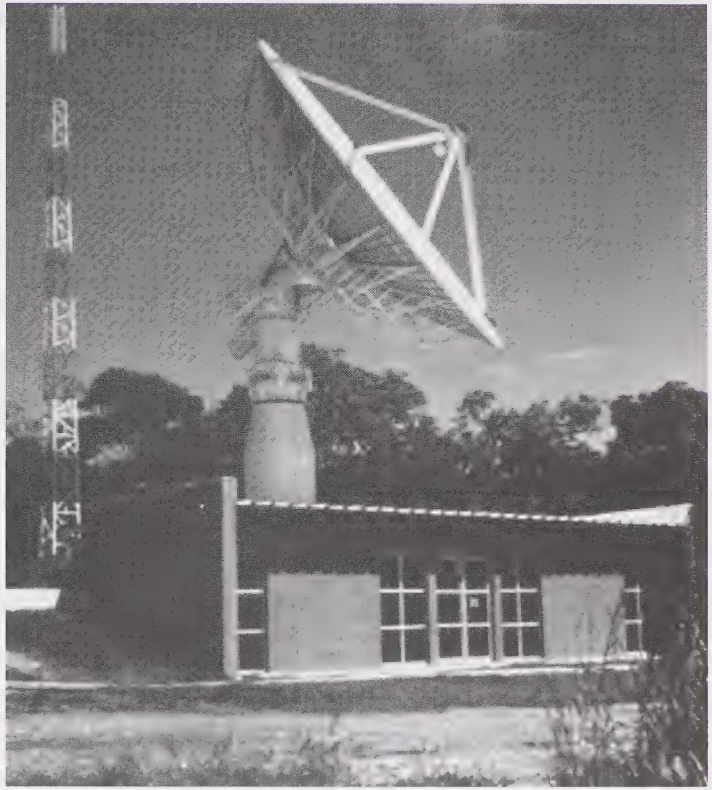
Assignment



You are now ready to complete the section assignment.

Module Summary

In this module you investigated the trigonometric functions and their graphs. In particular, you built on your knowledge of right-triangle trigonometry. The primary trigonometric functions—sine, cosine, and tangent—and their reciprocals—cosecant, secant, and cotangent—were defined in terms of angles drawn in standard position in the coordinate plane. A special device called the *unit circle* assisted you in determining the values of the trigonometric functions of these angles and aided you in developing relationships among the functions. As a result you were able to solve trigonometric equations, graph the trigonometric functions, and verify identities.


NASA

The radar station in the photograph uses sophisticated techniques to pinpoint distant aircraft. It is used to determine position and distance—problems that have concerned navigators, surveyors, and builders since ancient times. Trigonometry, with its origins in the measurement of triangles, has become a precise tool to support our modern technological society.

Final Module Assignment

Assignment Booklet

You are now ready to complete the final module assignment.

APPENDIX

Glossary

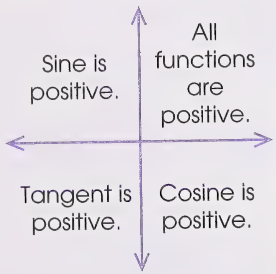
Suggested Answers

Glossary

Amplitude: the maximum vertical distance of the curve from a horizontal axis of symmetry

Asymptote: a line that a graph approaches but never touches

CAST rule: a rule which determines the signs of the primary trigonometric functions in relationship to their positions on the coordinate plane



Complementary angles: two angles whose sum is a right angle

Coterminal angles: angles which have the same terminal ray and initial ray

Domain: the set of all x-values for a specific function

Function: the set of all ordered pairs (x, y) in which each x is paired with one y

Period: the horizontal distance from any point to the next point where the cycle begins

Phase shift: the shifting of a graph to the right or left along the x-axis

Quadrantal angle: an angle in which the terminal arm coincides with an axis

Radian: the measure of the central angle of a circle that subtends or cuts off an arc equal in length to the radius of the circle

Range: the set of all y-values for a specific function

Reference angle: the angle formed between the terminal arm of a given angle and the x-axis

Sinusoidal: shaped like a sine curve

Terminal arm: an initial arm rotated through an angle θ to a terminal position

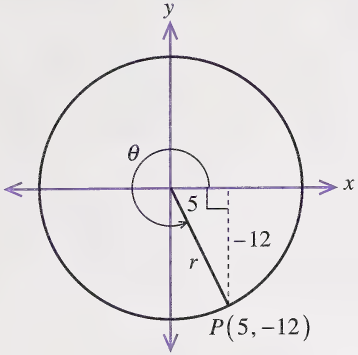
Transformation: a passage from one figure or expression to another

Vertical translation: the shifting of a graph up or down along the y-axis

Suggested Answers

Section 1: Activity 1

1. a.



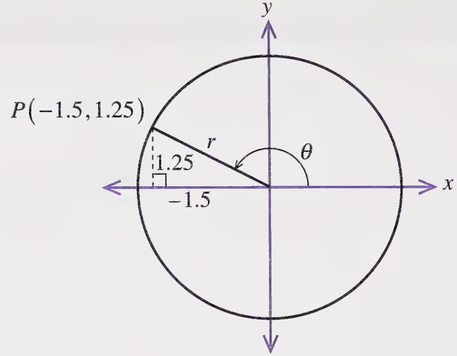
$$r = \sqrt{5^2 + (-12)^2}$$
$$= \sqrt{25 + 144}$$
$$= \sqrt{169}$$
$$= 13$$

$\sin\theta = \dfrac{-12}{13}$ $\csc\theta = \dfrac{13}{-12}$

$\cos\theta = \dfrac{5}{13}$ $\sec\theta = \dfrac{13}{5}$

$\tan\theta = \dfrac{-12}{5}$ $\cot\theta = \dfrac{5}{-12}$

b.



$$r = \sqrt{(-1.5)^2 + (1.25)^2}$$
$$= \sqrt{2.25 + 1.5625}$$
$$= \sqrt{3.8125}$$

$\sin\theta = \dfrac{1.25}{\sqrt{3.8125}}$
$\doteq 0.6402$

$\cos\theta = \dfrac{-1.5}{\sqrt{3.8125}}$
$\doteq -0.7682$

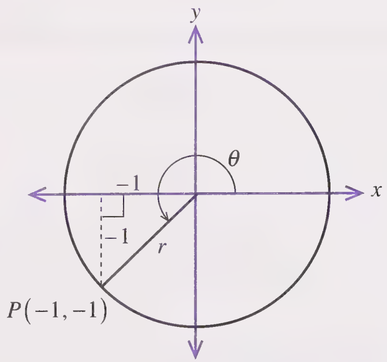
$\tan\theta = \dfrac{1.25}{-1.5}$
$\doteq -0.8333$

$\csc\theta \doteq 1.5620$

$\sec\theta \doteq -1.3017$

$\cot\theta = -1.2000$

c.



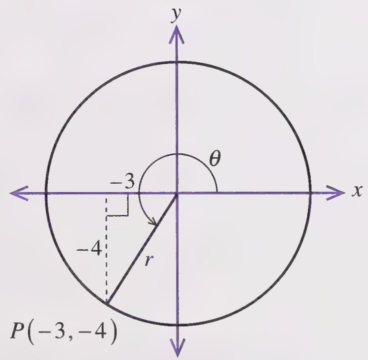
$$r=\sqrt{(-1)^2+(-1)^2}$$
$$=\sqrt{2}$$

$$\sin\theta=\frac{-1}{\sqrt{2}} =\frac{-\sqrt{2}}{2}$$

$$\cos\theta=\frac{-1}{\sqrt{2}} =\frac{-\sqrt{2}}{2}$$

$$\tan\theta=1$$
$$\csc\theta=-\sqrt{2}$$
$$\sec\theta=-\sqrt{2}$$
$$\cot\theta=1$$

2. a.

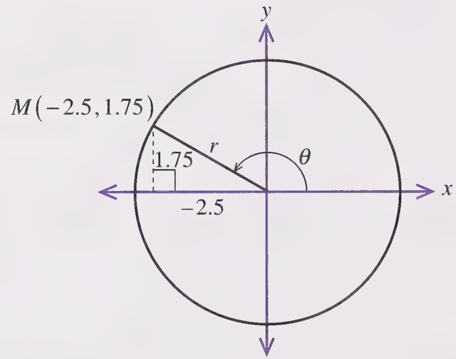


$$r=\sqrt{(-3)^2+(-4)^2}$$
$$=\sqrt{9+16}$$
$$=\sqrt{25}$$
$$=5$$

$$\sin\theta=\frac{-4}{5}\qquad \cos\theta=\frac{-3}{5}\qquad \tan\theta=\frac{-4}{-3}=\frac{4}{3}$$

$$\csc\theta=\frac{5}{-4}\qquad \sec\theta=\frac{5}{-3}\qquad \cot\theta=\frac{-3}{-4}=\frac{3}{4}$$

b.



$$\begin{aligned} r &= \sqrt{(-2.5)^2 + (1.75)^2} \\ &= \sqrt{6.25 + 3.0625} \\ &= \sqrt{9.3125} \end{aligned}$$

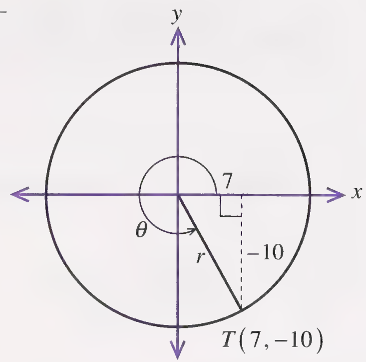
$$\sin\theta = \frac{1.75}{\sqrt{9.3125}} = 0.5735 \qquad \cos\theta = \frac{-2.5}{\sqrt{9.3125}} = -0.8192$$

$$\tan\theta = \frac{1.75}{-2.5} = -0.7000 \qquad \csc\theta = \frac{\sqrt{9.3125}}{1.75} = 1.7438$$

$$\sec\theta = \frac{\sqrt{9.3125}}{-2.5} = -1.2207 \qquad \cot\theta = \frac{-2.5}{1.75} = -1.4286$$

c.

$$\begin{aligned} r &= \sqrt{(7)^2 + (-10)^2} \\ &= \sqrt{49 + 100} \\ &= \sqrt{149} \end{aligned}$$

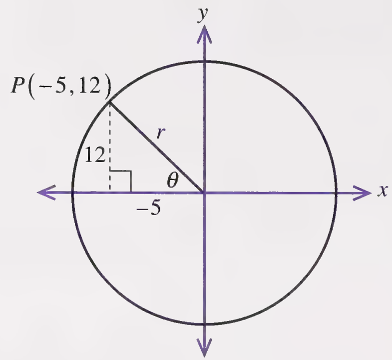


$$\sin\theta = \frac{-10}{\sqrt{149}} = -0.82 \qquad \cos\theta = \frac{7}{\sqrt{149}} = 0.57 \qquad \tan\theta = \frac{-10}{7} = -1.43$$

$$\csc\theta = \frac{\sqrt{149}}{-10} = -1.22 \qquad \sec\theta = \frac{\sqrt{149}}{7} = 1.74 \qquad \cot\theta = \frac{7}{-10} = -0.70$$

3.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r^2 &= (-5)^2 + (12)^2 \\ r^2 &= 25 + 144 \\ r^2 &= 169 \\ r &= 13 \end{aligned}$$



$$\sin\theta = \frac{12}{13} \qquad \cos\theta = \frac{-5}{13} \qquad \tan\theta = \frac{12}{-5}$$

$$\csc\theta = \frac{13}{12} \qquad \sec\theta = \frac{13}{-5} \qquad \cot\theta = \frac{-5}{12}$$

4. **a.** fourth quadrant **b.** first quadrant
c. third quadrant **d.** fourth quadrant
e. second quadrant

5. **a.** fourth quadrant **b.** second quadrant
c. first quadrant **d.** fourth quadrant
e. third quadrant

6. **a.** The terminal arm lies in the first and second quadrants.

b. $\sec\theta = \frac{1}{\cos\theta}$

Since $\cos\theta > 0$ in the first and fourth quadrants, $\sec\theta$ must be positive in the first and fourth quadrants.

c. $\cos\theta < 0$ in the second and third quadrants. $\csc\theta > 0$ in the first and second quadrants. Therefore, $\cos\theta < 0$ and $\csc\theta > 0$ in the second quadrant.

d. $\tan\theta < 0$ in the second and fourth quadrants. $\sec\theta > 0$ in the first and fourth quadrants. Therefore, $\tan\theta < 0$ and $\sec\theta > 0$ in the fourth quadrant.

7. **a.** The terminal arm lies in the second and fourth quadrants.

b. The terminal arm lies in the first and fourth quadrants since $\sec\theta = \frac{1}{\cos\theta}$ and $\cos\theta > 0$ in the first and fourth quadrants.

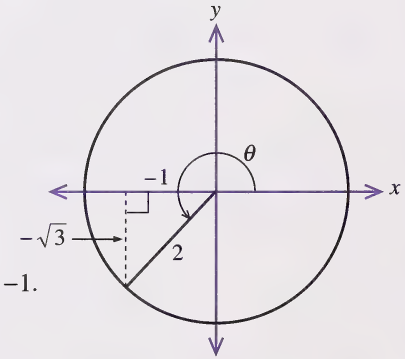
c. $\sin\theta < 0$ in the third and fourth quadrants.
$\cot\theta > 0$ in the first and third quadrants.
Therefore, $\sin\theta < 0$ and $\cot\theta > 0$ in the third quadrant.

d. $\cos\theta > 0$ in the first and fourth quadrants.
$\sec\theta > 0$ in the first and fourth quadrants.
Therefore, $\cos\theta > 0$ and $\sec\theta > 0$ in the first and fourth quadrants.

Section 1: Activity 2

1. Since $\sin\theta = \frac{-\sqrt{3}}{2}$ and $\sin\theta = \frac{y}{r}$, then $y = -\sqrt{3}$ and $r = 2$, where $180° < \theta < 270°$.

$$r^2 = x^2 + y^2$$
$$2^2 = x^2 + \left(-\sqrt{3}\right)^2$$
$$4 = x^2 + 3$$
$$1 = x^2$$
$$\pm 1 = x$$



In the third quadrant, $x = -1$.

$$\cos\theta = \frac{-1}{2}$$

$$\tan\theta = \sqrt{3}$$

$$\csc\theta = \frac{2}{-\sqrt{3}} = \frac{2\sqrt{3}}{-3}$$

$$\sec\theta = -2$$

$$\cot\theta = \frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Remember to always rationalize the denominator.

$$\frac{2}{-\sqrt{3}} = \frac{2}{-\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{-3}$$

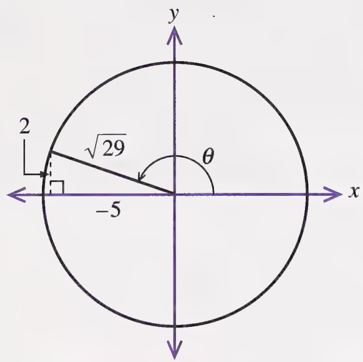
b. $\sin\theta = \frac{2}{\sqrt{29}} = \frac{2\sqrt{29}}{29}$ $\cos\theta = \frac{-5}{\sqrt{29}} = \frac{-5\sqrt{29}}{29}$ $\tan\theta = \frac{2}{-5}$

$\csc\theta = \frac{\sqrt{29}}{2}$ $\sec\theta = \frac{\sqrt{29}}{-5}$ $\cot\theta = \frac{-5}{2}$

2. a. $\sec\theta = \frac{\sqrt{29}}{-5}$

$r = \sqrt{29}$ and $x = -5$

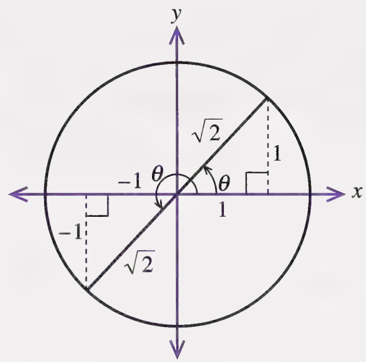
$$r^2 = x^2 + y^2$$
$$\left(\sqrt{29}\right)^2 = (-5)^2 + y^2$$
$$29 = 25 + y^2$$
$$4 = y^2$$
$$\pm 2 = y$$



In the second quadrant, $y = 2$.

3. $\tan\theta = 1 = \frac{1}{1}$ or $\frac{-1}{-1}$

$\tan\theta > 0$ in the first and third quadrants.



$$r = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

or

$$r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

In the first quadrant, the values of the other trigonometric ratios are as follows:

$$\sin\theta = \frac{1}{\sqrt{2}} \doteq 0.7071 \qquad \cos\theta = \frac{1}{\sqrt{2}} \doteq 0.7071$$

$$\sec\theta = \sqrt{2} \doteq 1.4142 \qquad \csc\theta = \sqrt{2} \doteq 1.4142$$

$$\cot\theta = 1.0000$$

In the third quadrant, the values of the other trigonometric ratios are as follows:

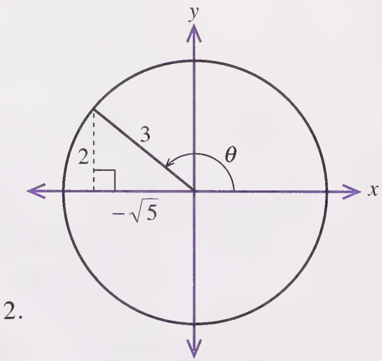
$$\sin\theta = \frac{-1}{\sqrt{2}} \doteq -0.7071 \qquad \cos\theta = \frac{-1}{\sqrt{2}} \doteq -0.7071$$

$$\sec\theta = -\sqrt{2} \doteq -1.4142 \qquad \csc\theta = -\sqrt{2} \doteq -1.4142$$

$$\cot\theta = 1.0000$$

4. Since $\cos\theta = \frac{-\sqrt{5}}{3}$ and $\cos\theta = \frac{x}{r}$, then $x = -\sqrt{5}$ and $r = 3$, where $90° < \theta < 180°$.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ (3)^2 &= \left(-\sqrt{5}\right)^2 + y^2 \\ 9 &= 5 + y^2 \\ 4 &= y^2 \\ y &= \pm 2 \end{aligned}$$



In the second quadrant, $y = 2$.

The other trigonometric ratios are as follows:

$$\sin\theta = \frac{2}{3} \qquad \csc\theta = \frac{3}{2}$$

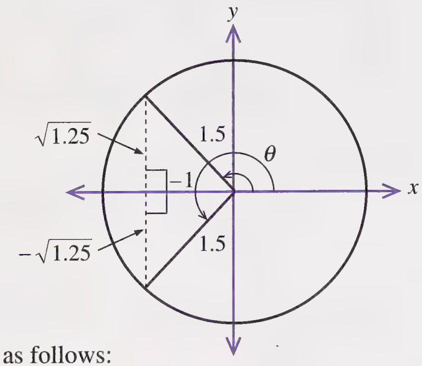
$$\tan\theta = \frac{2}{-\sqrt{5}} = \frac{2\sqrt{5}}{-5} \qquad \sec\theta = \frac{3}{-\sqrt{5}} = \frac{3\sqrt{5}}{-5}$$

Remember to rationalize the denominator.

$$\cot\theta = \frac{-\sqrt{5}}{2}$$

5. Since $\sec\,\theta = -1.5$, then $\sec\,\theta = \frac{r}{x} = \frac{1.5}{-1}$. Therefore, $r = 1.5$ and $x = -1$. In the second and third quadrants $\sec\,\theta$ is negative for $0° < \theta < 270°$.

$$\begin{aligned} r^2 &= x^2 + y^2 \\ (1.5)^2 &= (-1)^2 + y^2 \\ y^2 &= 2.25 - 1 \\ y^2 &= 1.25 \\ y &= \pm\sqrt{1.25} \end{aligned}$$



In the second quadrant, the values of the other trigonometric ratios are as follows:

$$\sin\,\theta = \frac{\sqrt{1.25}}{1.5} = 0.7454 \qquad \cos\,\theta = \frac{-1}{1.5} = -0.6667$$

$$\tan\,\theta = \frac{\sqrt{1.25}}{-1} = -1.1180 \qquad \csc\,\theta = \frac{1.5}{\sqrt{1.25}} = 1.3416$$

$$\cot\,\theta = \frac{-1}{\sqrt{1.25}} = -0.8944$$

In the third quadrant, the values of the other trigonometric ratios are as follows:

$$\sin\,\theta = \frac{-\sqrt{1.25}}{1.5} = -0.7454 \qquad \cos\,\theta = \frac{-1}{1.5} = -0.6667$$

$$\tan\,\theta = \frac{-\sqrt{1.25}}{-1} = 1.1180 \qquad \csc\,\theta = \frac{1.5}{-\sqrt{1.25}} = -1.3416$$

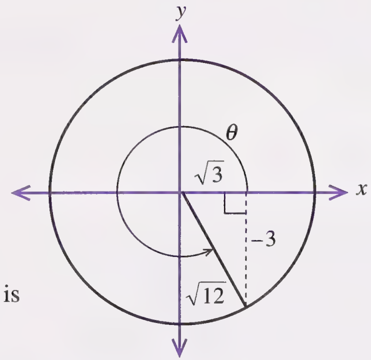
$$\cot\,\theta = \frac{-1}{-\sqrt{1.25}} = 0.8944$$

6. $3\cot\,\theta = -\sqrt{3}$

$$\cot\,\theta = \frac{\sqrt{3}}{-3}$$

Since $\cot\,\theta = \frac{x}{y}$, then $x = \sqrt{3}$ and $y = -3$.

$$r^2 = x^2 + y^2$$
$$r^2 = \left(\sqrt{3}\right)^2 + (-3)^2$$
$$r^2 = 3 + 9$$
$$r^2 = 12$$
$$r = \pm\sqrt{12}$$



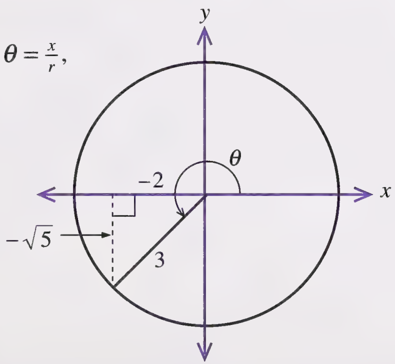
In the fourth quadrant, $\sin\theta$ is negative.

$$\therefore\ \sin\theta = \frac{y}{r}$$
$$= \frac{-3}{\sqrt{12}}$$
$$\doteq -0.87$$

7. Since $\cos\theta$ is negative and $\tan\theta$ is positive, $\csc\theta$ must be in the third quadrant.

Since $\cos\theta = \frac{-2}{3}$ and $\cos\theta = \frac{x}{r}$, then $x = -2$ and $r = 3$.

$$r^2 = x^2 + y^2$$
$$3^2 = (-2)^2 + y^2$$
$$9 = 4 + y^2$$
$$y^2 = 5$$
$$y = \pm\sqrt{5}$$

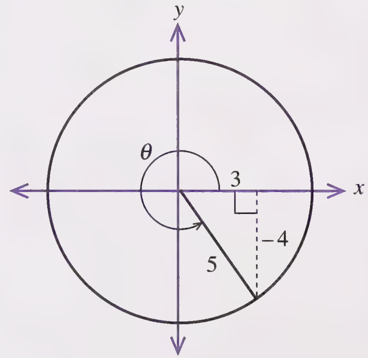


In the third quadrant, y is negative.

$$\therefore\ \csc\theta = \frac{3}{-\sqrt{5}}$$
$$= \frac{3\sqrt{5}}{-5}$$

8. **a.** Since $\tan\theta = \frac{-4}{3}$ and $\tan\theta = \frac{y}{x}$, then $y = -4$ and $x = 3$.

$$r^2 = x^2 + y^2$$
$$r^2 = (3)^2 + (-4)^2$$
$$r^2 = 9 + 16$$
$$r^2 = 25$$
$$r = 5$$



b. $\sin\theta = \frac{-4}{5}$ $\qquad$ $\cos\theta = \frac{3}{5}$

$\csc\theta = \frac{5}{-4}$ $\qquad$ $\cot\theta = \frac{3}{-4}$

$\sec\theta = \frac{5}{3}$

Section 1: Activity 3

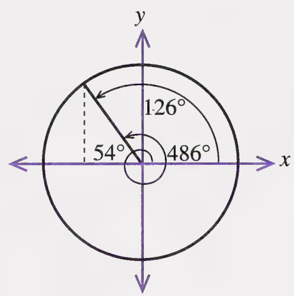
1. a. The angle terminates in the second quadrant.

$$\begin{aligned} R &= 180° - 169° \\ &= 11° \end{aligned}$$

b. The angle terminates in the fourth quadrant.

$$\begin{aligned} R &= 360° - 343° \\ &= 17° \end{aligned}$$

c. The angle between 0° and 360° coterminal with 486° is $486° - 360° = 126°$. Therefore, the angle terminates in the second quadrant.

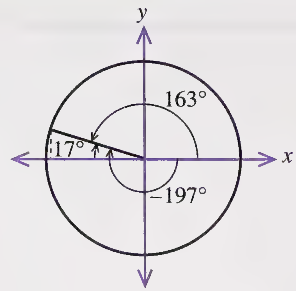


$$\begin{aligned} R &= 180° - 126° \\ &= 54° \end{aligned}$$

d. The angle terminates in the third quadrant.

$$\begin{aligned} R &= 256° - 180° \\ &= 76° \end{aligned}$$

e. The angle between 0° and 360° coterminal with −197° is $360° - 197° = 163°$. Therefore, the angle terminates in the second quadrant.



$$\begin{aligned} R &= 180° - 163° \\ &= 17° \end{aligned}$$

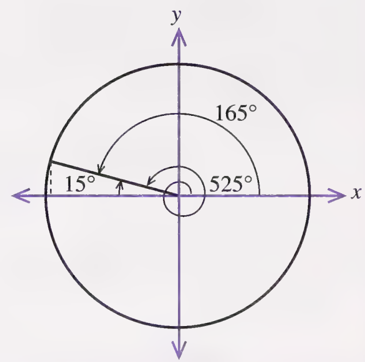
f. The angle terminates in the second quadrant.

$$\begin{aligned} R &= 180° - 155° \\ &= 25° \end{aligned}$$

g. The angle terminates in the third quadrant.

$$\begin{aligned} R &= 180° - 120° \\ &= 60° \end{aligned}$$

h. The angle between 0° and 360° coterminal with 525° is $525° - 360° = 165°$. Therefore, the angle terminates in the second quadrant.

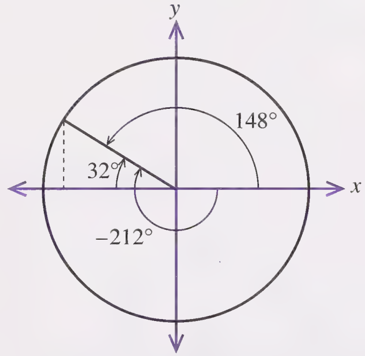


$$\begin{aligned} R &= 180° - 165° \\ &= 15° \end{aligned}$$

i. The angle terminates in the third quadrant.

$R = 246° - 180°$
$= 66°$

j. The angle between 0° and 360° coterminal with $-212°$ is $360° - 212° = 148°$. Therefore, the angle terminates in the second quadrant.



$R = 180° - 148°$
$= 32°$

2. a. $\sin 178° = +\sin 2°$
An angle of 178° is in the second quadrant.

b. $\cos 415° = +\cos 55°$
$415° - 360° = 55°$
An angle of 55° is in the first quadrant.

c. $\sec 193° = -\sec 13°$
An angle of 193° is in the third quadrant.

d. $\tan 545° = +\tan 185°$
$545° - 360° = 185°$
An angle of 185° is in the third quadrant.

e. $\sin 288° = -\sin 72°$
An angle of 288° is in the fourth quadrant.

f. $\csc 125° = +\csc 55°$
An angle of $\csc 125°$ is in the second quadrant.

g. $\tan 212° = +\tan 32°$
An angle of $\tan 212°$ is in the third quadrant.

h. $\cos(-190°) = -\cos(-10°)$
An angle of $-190°$ is in the second quadrant.

3. a. $\sin(360° - \theta) = -\sin\theta$
b. $\cos(360° - \theta) = +\cos\theta$
c. $\tan(360° - \theta) = -\tan\theta$
d. $\csc(360° - \theta) = -\csc\theta$
e. $\sec(360° - \theta) = +\sec\theta$
f. $\cot(360° - \theta) = -\cot\theta$

4. a. $\sin(360° - \theta) = -\sin\theta$
b. $\tan(180° + \theta) = +\tan\theta$
c. $\csc(180° - \theta) = +\csc\theta$
d. $\cot(360° - \theta) = -\cot\theta$
e. $\sec(180 + \theta) = -\sec\theta$
f. $\cos(360° - \theta) = +\cos\theta$

5. a. $\tan 150° = \tan(180° - 150°)$

$= -\tan 30°$ (In the second quadrant, tangent is negative.)

$= \frac{-1}{\sqrt{3}}$

$= \frac{-1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{-\sqrt{3}}{3}$

The denominator of a fraction is usually in nonsurd (nonroot) form.

b. $\cos 300° = \cos(360° - 300°)$

$= \cos 60°$ (In the fourth quadrant, cosine is positive.)

$= \frac{1}{2}$

c. $\sin(-225°) = \sin(360° - 225°)$

$= \sin 135°$

$= \sin(180° - 135°)$

$= \sin 45°$ (In the second quadrant, sine is positive.)

$= \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{2}}{2}$

$-225°$ is in the second quadrant.

d. $\cos(-315°) = \cos(360° - 315°)$

$= \cos 45°$ (In the first quadrant, cosine is positive.)

$= \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{2}}{2}$

e. $\sin 210° = \sin(210° - 180°)$

$= -\sin 30°$ (In the third quadrant, sine is negative.)

$= -\frac{1}{2}$

f. $\tan(-300°) = \tan(360° - 300°)$

$= \tan 60°$ (In the first quadrant, tangent is positive.)

$= \frac{\sqrt{3}}{1}$

$= \sqrt{3}$

g. $\cos(-135°) = \cos(180° - 135°)$

$= -\cos 45°$ (In the third quadrant, cosine is negative.)

$= -\frac{\sqrt{2}}{2}$

h. $\sec 225° = \sec(225° - 180°)$

$= -\sec 45°$ (In the third quadrant, secant is negative.)

$= -\sqrt{2}$

6. a. $(\sin 315°)^2 + (\cos 45°)^2 = (-\sin 45°)^2 + (\cos 45°)^2$

$= \left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{2} + \frac{1}{2}$

$= 1$

b. $2\sec(-45°) - \cot 135° = 2\sec(-45°) - (-\cot 45°)$

$$= 2\left(\sqrt{2}\right) - (-1)$$
$$= 2\sqrt{2} + 1$$

c. $\sin 225° \times \cos(-60°) = (-\sin 45°) \times \cos(-60°)$

$$= -\frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{-1}{2\sqrt{2}}$$
$$= \frac{-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{-\sqrt{2}}{4}$$

d. $2\tan 45° - 6\sec(-240°) = 2\tan 45° - 6\sec 120°$

$$= 2\tan 45° - 6(-\sec 60°)$$
$$2(1) - 6\left(-\frac{2}{1}\right)$$
$$= 2 + 12$$
$$= 14$$

Section 1: Follow-up Activities

Extra Help

1. **a.** Coterminal angles

b. clockwise

2. **a.** **fourth** quadrant

$R = 360° - \theta$
$= \mathbf{80°}$

b. **second** quadrant

$R = \mathbf{180° - 140°}$
$= \mathbf{40°}$

c. **third** quadrant

$R = \mathbf{210° - 180°}$
$= \mathbf{30°}$

d. $360° - 120° = 240°$

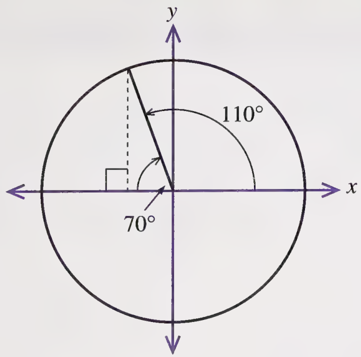
third quadrant
$R = \mathbf{240° - 180°}$
$= \mathbf{60°}$

e. $360° - 190° = 170°$

second quadrant
$R = \mathbf{180° - 170°}$
$= \mathbf{10°}$

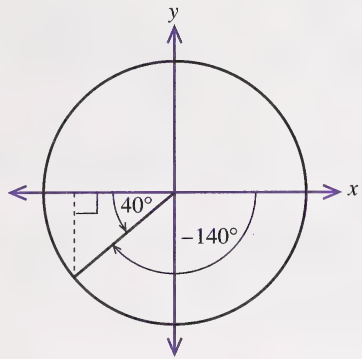
3. **a.** second quadrant
b. first quadrant
c. third quadrant
d. fourth quadrant
e. fourth quadrant

4. a.



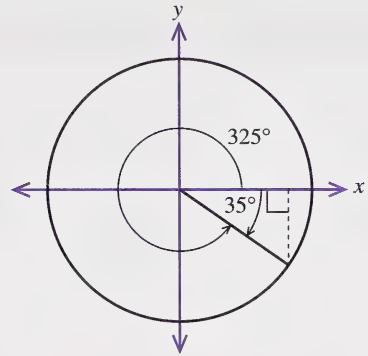
$$\sin 110° = \sin 70° = 0.9397$$

b.



$$\tan(-140°) = \tan 40° = 0.8391$$

c.

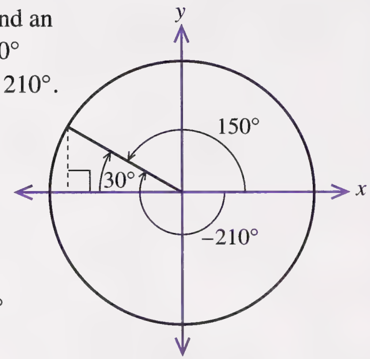


$$\sec 325° = \sec 35° = 1.2208$$

5. $360° - 30° = 330°$

Enrichment

1. a. Add 360° to −210° to find an angle between 0° and 360° that is coterminal with −210°.

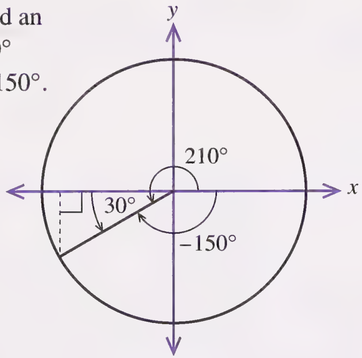


$$360° - 210° = 150°$$

$$R = 180° - 150° = 30°$$

$$\sec(-210°) = -\sec 30° = -\frac{2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$$

b. Add 360° to $-150°$ to find an angle between 0° and 360° that is coterminal with $-150°$.



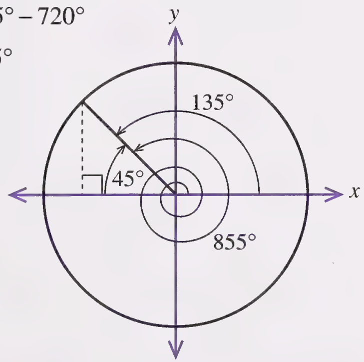
$$360° - 150° = 210°$$

$$\begin{aligned} R &= 210° - 180° \\ &= 30° \end{aligned}$$

$$\begin{aligned} \cot(-150°) &= \cot(30°) \\ &= \sqrt{3} \end{aligned}$$

c. The angle between 0° and 360° coterminal with 855° is as follows:

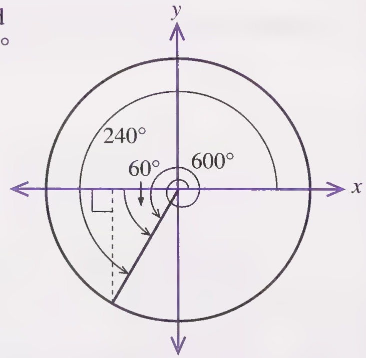
$$\begin{aligned} 855° - 2(360°) &= 855° - 720° \\ &= 135° \end{aligned}$$



$$\begin{aligned} R &= 180° - 135° \\ &= 45° \end{aligned}$$

$$\begin{aligned} \cos 855° &= -\cos 45° \\ &= -\frac{1}{\sqrt{2}} \\ &= -\frac{\sqrt{2}}{2} \end{aligned}$$

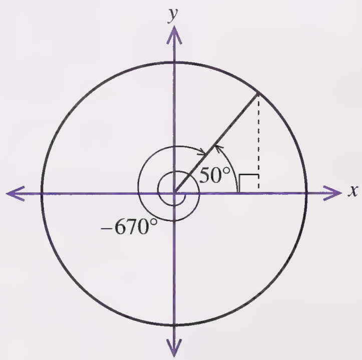
d. The angle between 0° and 360° coterminal with 600° is $600° - 360° = 240°$.



$$\begin{aligned} R &= 240° - 180° \\ &= 60° \end{aligned}$$

$$\begin{aligned} \csc 600° &= -\csc 60° \\ &= -\frac{2}{\sqrt{3}} \\ &= \frac{-2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ &= \frac{-2\sqrt{3}}{3} \end{aligned}$$

2. Find the angle between 0° and 360°. Add $2(360°)$ to $-670°$.



$$720° - 670° = 50°$$

Thus, 50° is coterminal with $-670°$. Since 50° is an acute angle, the reference angle is 50°.

$$\begin{aligned} \sin(-670°) &= \sin 50° \\ &\doteq 0.766 \end{aligned}$$

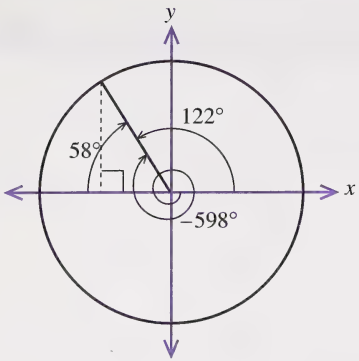
3. Find the angle between 0° and 360°. Add $2(360°)$ to $-598°$.

$$720° - 598° = 122°$$

Thus, 122° is coterminal with $-598°$.

$$\begin{aligned} R &= 180° - 122° \\ &= 58° \end{aligned}$$

$$\begin{aligned} \cos(-598°) &= -\cos 58° \\ &= -0.5299 \end{aligned}$$



Section 2: Activity 1

1. a. $(\cos 32°, \sin 32°) \doteq (0.9, 0.5)$

   b. $(\cos 287°, \sin 287°) \doteq (0.3, -1.0)$

   c. $(\cos(-193°), \sin(-193°)) \doteq (-1.0, 0.2)$

2. a. $\sin\theta = -\frac{5}{13} = y$

$$\begin{aligned} x^2 + y^2 &= 1 \\ x^2 + \left(-\frac{5}{13}\right)^2 &= 1 \\ x^2 + \frac{25}{169} &= 1 \\ x^2 &= 1 - \frac{25}{169} \\ x^2 &= \frac{169}{169} - \frac{25}{169} \\ x^2 &= \frac{144}{169} \\ x &= \pm\sqrt{\frac{144}{169}} \\ &= \pm\frac{12}{13} \end{aligned}$$

Since θ terminates in the fourth quadrant, then $x = \frac{12}{13}$.

$\therefore \sin\theta = -\frac{5}{13}$ $\csc\theta = -\frac{13}{5}$

$\cos\theta = \frac{12}{13}$ $\sec\theta = \frac{13}{12}$

$\tan\theta = \frac{y}{x} = -\frac{5}{12}$ $\cot\theta = \frac{x}{y} = -\frac{12}{5}$

b. $\cos\theta = \frac{2}{3} = x$

$$x^2 + y^2 = 1$$
$$\left(\frac{2}{3}\right)^2 + y^2 = 1$$
$$\frac{4}{9} + y^2 = 1$$
$$y^2 = 1 - \frac{4}{9}$$
$$y^2 = \frac{9}{9} - \frac{4}{9}$$
$$y^2 = \frac{5}{9}$$
$$y = \pm\sqrt{\frac{5}{9}}$$
$$= \pm\frac{\sqrt{5}}{3}$$

Since θ terminates in the first quadrant, then $y = \frac{\sqrt{5}}{3}$.

$\therefore\ \sin\theta = \frac{\sqrt{5}}{3}$ $\qquad \cos\theta = \frac{2}{3}$ $\qquad \tan\theta = \frac{y}{x} = \frac{\sqrt{5}}{2}$

$\csc\theta = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$ $\qquad \sec\theta = \frac{3}{2}$ $\qquad \cot\theta = \frac{x}{y} = \frac{2\sqrt{5}}{5}$

c. $\sec\theta = -2$

$$\cos\theta = \frac{1}{\sec\theta}$$
$$x = -\frac{1}{2}$$

$$x^2 + y^2 = 1$$
$$\left(-\frac{1}{2}\right)^2 + y^2 = 1$$
$$\frac{1}{4} + y^2 = 1$$
$$y^2 = 1 - \frac{1}{4}$$
$$y^2 = \frac{3}{4}$$
$$y = \pm\sqrt{\frac{3}{4}}$$
$$= \pm\frac{\sqrt{3}}{2}$$

Since θ terminates in the second quadrant, then $y = \frac{\sqrt{3}}{2}$.

$\therefore\ \sin\theta = \frac{\sqrt{3}}{2}$ $\qquad \csc\theta = \frac{2\sqrt{3}}{3}$

$\cos\theta = -\frac{1}{2}$ $\qquad \sec\theta = -2$

$\tan\theta = \frac{y}{x} = -\sqrt{3}$ $\qquad \cot\theta = \frac{x}{y} = -\frac{\sqrt{3}}{3}$

d. $\csc\theta = -\frac{4}{3}$

$\sin\theta = \frac{1}{\csc\theta}$

$= \frac{1}{-\frac{4}{3}}$

$= -\frac{3}{4}$

$$x^2 + y^2 = 1$$
$$x^2 + \left(-\frac{3}{4}\right)^2 = 1$$
$$x^2 + \frac{9}{16} = 1$$
$$x^2 = 1 - \frac{9}{16}$$
$$x^2 = \frac{7}{16}$$
$$x = \pm\sqrt{\frac{7}{16}}$$
$$= \pm\frac{\sqrt{7}}{4}$$

Since θ terminates in the third quadrant, then $x = -\frac{\sqrt{7}}{4}$.

$\therefore\ \sin\theta = -\frac{3}{4}$ $\qquad \csc\theta = -\frac{4}{3}$

$\cos\theta = -\frac{\sqrt{7}}{4}$ $\qquad \sec\theta = -\frac{4\sqrt{7}}{7}$

$\tan\theta = \frac{y}{x} = \frac{3\sqrt{7}}{7}$ $\qquad \cot\theta = \frac{x}{y} = \frac{\sqrt{7}}{3}$

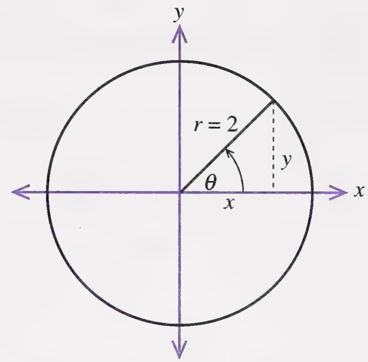
3.

Quadrantal Angle θ	0°	90°	180°	270°	360°
Point on Terminal Arm	$(1, 0)$	$(0, 1)$	$(-1, 0)$	$(0, -1)$	$(1, 0)$
cos θ	1	0	−1	0	1
sin θ	0	1	0	−1	0
tan θ	0	undefined	0	undefined	0
sec θ	1	undefined	−1	undefined	1
csc θ	undefined	1	undefined	−1	undefined
cot θ	undefined	0	undefined	0	undefined

undefined

4. **a.** −1 **b.** 0 **c.** 1

5.



No, you will not have the same representation.

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{2} \qquad \sin\theta = \frac{y}{2}$$

$$x = 2\cos\theta \qquad y = 2\sin\theta$$

Section 2: Activity 2

1. a.
$$\begin{aligned} \theta &= \frac{a}{r} \\ 3.6 &= \frac{a}{7} \\ a &= 7 \times 3.6 \\ &= 25.2 \end{aligned}$$

b.
$$\begin{aligned} \theta &= \frac{a}{r} \\ 1.2 &= \frac{60}{r} \\ 1.2r &= 60 \\ r &= \frac{60}{1.2} \\ &= 50 \end{aligned}$$

c.
$$\begin{aligned} \theta &= \frac{a}{r} \\ &= \frac{36}{16} \\ &= 2.25 \end{aligned}$$

2.
$$\begin{aligned} \theta &= \frac{a}{r} \\ 2 &\doteq \frac{a}{6400} \\ a &\doteq 2 \times 6400 \\ &\doteq 12\,800 \text{ km} \end{aligned}$$

The distance at the equator is about $12\,800$ km.

3.
$$\begin{aligned} \theta &= \frac{a}{r} \\ \frac{\pi}{2} &= \frac{10}{r} \\ \frac{\pi}{2}r &= 10 \\ r &= \frac{10}{\frac{\pi}{2}} \\ &= \frac{20}{\pi} \\ &\doteq 6.4 \text{ cm} \end{aligned}$$

The radius of the circle is approximately 6.4 cm.

4.
$$\begin{aligned} \theta &= \frac{a}{r} \\ \theta &= \frac{12\,400}{7500} \\ &\doteq 1.65 \end{aligned}$$

The radian measure is approximately 1.65.

5. a.
$$\begin{aligned} \theta &= \frac{a}{r} \\ 2.5 &= \frac{a}{45} \\ a &= 2.5 \times 45 \\ &= 112.5 \text{ cm} \end{aligned}$$

The distance travelled is 112.5 cm.

b. The length of the spoke (r) is an approximate value because it joins the hub to the rim. It is not taken from the centre of the wheel.

6. a.

$$\theta = \frac{a}{r}$$

$$1.25 = \frac{a}{25}$$

$$a = 1.25 \times 25$$

$$= 31.25 \text{ cm}$$

The length of the arc is 31.25 cm.

b. The pendulum in a time clock will have a constant angle and arc length.

A pendulum which is activated mechanically will slow down and eventually stop. Therefore, the angle and arc length must change.

Section 2: Activity 3

1. a. $1^\circ = \frac{\pi}{180^\circ}$

$$\therefore\ 45^\circ = \frac{\pi}{180^\circ} \times 45^\circ$$

$$= \frac{\pi}{4}$$

$$\doteq 0.79 \text{ rad}$$

b. $1^\circ = \frac{\pi}{180^\circ}$

$$\therefore\ -75^\circ = \frac{\pi}{180^\circ} \times (-75^\circ)$$

$$= -\frac{5\pi}{12}$$

$$\doteq -1.31 \text{ rad}$$

c. $1^\circ = \frac{\pi}{180^\circ}$

$$\therefore\ 288^\circ = \frac{\pi}{180^\circ} \times 288^\circ$$

$$= \frac{8\pi}{5}$$

$$\doteq 5.03 \text{ rad}$$

d. $1^\circ = \frac{\pi}{180^\circ}$

$$\doteq 0.02 \text{ rad}$$

2. a. $1 \text{ rad} = \frac{180^\circ}{\pi}$

$$\therefore\ \frac{\pi}{3} \text{ rad} = \frac{180^\circ}{\pi} \times \frac{\pi}{3}$$

$$= 60.0^\circ$$

b. $1 \text{ rad} = \frac{180^\circ}{\pi}$

$$\therefore\ -\frac{5\pi}{12} \text{ rad} = \frac{180^\circ}{\pi} \times \left(-\frac{5\pi}{12}\right)$$

$$= -75.0^\circ$$

c. $1 \text{ rad} = \frac{180^\circ}{\pi}$

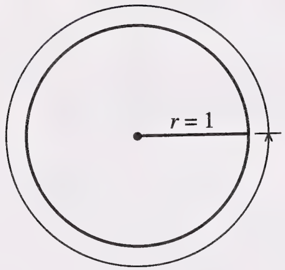
$$\therefore\ -2.7 \text{ rad} = \frac{180^\circ}{\pi} \times (-2.7)$$

$$= -154.7^\circ$$

d. $1 \text{ rad} = \frac{180^\circ}{\pi}$

$$\therefore\ 5 \text{ rad} = \frac{180°}{\pi} \times 5$$
$$= 286.5°$$

3. The radius of the circle is 1 cm.



$$C = 2\pi r$$
$$= 2\pi(1)$$
$$= 2\pi \text{ cm}$$

One complete revolution is equal to the circumference $(2\pi \text{ cm})$.

$$\therefore\ 2\frac{1}{4} \text{ revolutions} = 2\pi \times 2\frac{1}{4}$$
$$= 2\pi \times \frac{9}{4}$$
$$= \frac{9\pi}{2}$$

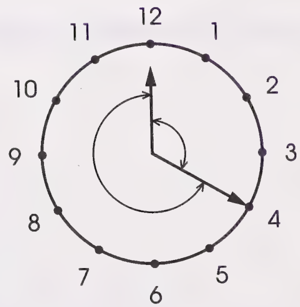
The disk travelled $\frac{9\pi}{2}$ cm.

4. $\theta = \frac{16}{5}$
$= 3.2$ rad

$$1 \text{ rad} = \frac{180°}{\pi}$$

$$\therefore\ 3.2 \text{ rad} = \frac{180°}{\pi} \times 3.2$$
$$\doteq 183.3465°$$

5. There are two measures: the obtuse angle and the reflex angle.



Obtuse Angle

$1 \text{ revolution} = 60 \text{ min}$

$$20 \text{ min} = \frac{20}{60} \text{ revolution}$$
$$= \frac{1}{3} \text{ revolution}$$

Reflex Angle

$360° - 120° = 240°$

$$1° = \frac{\pi}{180°}$$

$$\therefore\ 240° = \frac{\pi}{180°} \times 240°$$
$$= \frac{4\pi}{3} \text{ rad}$$

The central angle for 1 revolution is 360°. Thus, for $\frac{1}{3}$ revolution the central angle is $360° \times \frac{1}{3} = 120°$.

$$1° = \frac{\pi}{180°}$$

$$120° = \frac{\pi}{180°} \times 120°$$
$$= \frac{2\pi}{3} \text{ rad}$$

6. a. $\left(\cos\frac{4\pi}{3}, \sin\frac{4\pi}{3}\right) \doteq (-0.50, -0.87)$

b. $\left(\cos\frac{7\pi}{12}, \sin\frac{7\pi}{12}\right) \doteq (-0.26, 0.97)$

c. $\left(\cos -\frac{5\pi}{7}, \sin -\frac{5\pi}{7}\right) \doteq (-0.62, -0.78)$

d. $[\cos(1 \text{ rad}), \sin(1 \text{ rad})] \doteq (0.54, 0.84)$

Section 2: Activity 4

1.

Angle Measure (degrees)	Angle Measure (radian)	Coordinates
0	0	$(1, 0)$
30	$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
45	$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
60	$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
90	$\frac{\pi}{2}$	$(0, 1)$

2.

Angle Measure (rad)		Angle Measure (degrees)	Coordinates
	0	0	$(1, 0)$
First Quadrant	$\frac{\pi}{6}$	30	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
	$\frac{\pi}{4}$	45	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
	$\frac{\pi}{3}$	60	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
	$\frac{\pi}{2}$	90	$(0, 1)$
Second Quadrant	$\frac{2\pi}{3}$	120	$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
	$\frac{3\pi}{4}$	135	$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
	$\frac{5\pi}{6}$	150	$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
	π	180	$(-1, 0)$
Third Quadrant	$\frac{7\pi}{6}$	210	$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
	$\frac{5\pi}{4}$	225	$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

	$\frac{4\pi}{3}$	240	$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
	$\frac{3\pi}{2}$	270	$(0, -1)$
Fourth Quadrant	$\frac{5\pi}{3}$	300	$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
	$\frac{7\pi}{4}$	315	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
	$\frac{11\pi}{6}$	330	$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
	2π	360	$(1, 0)$

c. $\csc(-300°) = \dfrac{1}{\sin(-300°)}$
$= \dfrac{1}{\frac{\sqrt{3}}{2}}$
$= \dfrac{2\sqrt{3}}{3}$

d. $\cos 135° = -\dfrac{\sqrt{2}}{2}$

3. a. $\sin 120° = \dfrac{\sqrt{3}}{2}$

b. $\tan(-240°) = \dfrac{\sin(-240°)}{\cos(-240°)}$
$= \dfrac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$
$= \dfrac{\sqrt{3}}{2} \times \left(-\dfrac{2}{1}\right)$
$= -\sqrt{3}$

4. a. $\sin \dfrac{4\pi}{3} = -\dfrac{\sqrt{3}}{2}$

b. $\sec \dfrac{7\pi}{6} = \dfrac{1}{\cos \frac{7\pi}{6}}$
$= \dfrac{1}{-\frac{\sqrt{3}}{2}}$
$= -\dfrac{2\sqrt{3}}{3}$

c. $\cot\left(-\dfrac{11\pi}{6}\right) = \dfrac{\cos\left(-\frac{11\pi}{6}\right)}{\sin\left(-\frac{11\pi}{6}\right)}$
$= \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$
$= \dfrac{\sqrt{3}}{2} \times \dfrac{2}{1}$
$= \sqrt{3}$

d. $\cos \pi = -1$

e. $\csc\left(-\frac{7\pi}{4}\right)=\frac{1}{\sin\left(-\frac{7\pi}{4}\right)}$

$$=\frac{1}{\frac{\sqrt{2}}{2}}$$

$$=\frac{2\sqrt{2}}{2}$$

$$=\sqrt{2}$$

5. a. $\sin 225^\circ=-\frac{\sqrt{2}}{2}$

b. $\tan\frac{5\pi}{6}=\frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$

$$=\frac{1}{2}\times\left(-\frac{2}{\sqrt{3}}\right)$$

$$=-\frac{1}{\sqrt{3}}$$

$$=-\frac{\sqrt{3}}{3}$$

c. $\csc(-45^\circ)=\frac{1}{-\frac{\sqrt{2}}{2}}$

$$=-\frac{2}{\sqrt{2}}$$

$$=-\sqrt{2}$$

d. $\tan\left(-\frac{\pi}{3}\right)=\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$$=-\frac{\sqrt{3}}{2}\times\frac{2}{1}$$

$$=-\sqrt{3}$$

e. $\sec 300^\circ=\frac{1}{\frac{1}{2}}$

$$=2$$

f. $\sec(-210^\circ)=\frac{1}{-\frac{\sqrt{3}}{2}}$

$$=-\frac{2}{\sqrt{3}}$$

$$=-\frac{2\sqrt{3}}{3}$$

g. $\cos\left(-\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}$

h. $\cot\frac{5\pi}{4}=\frac{\cos\frac{5\pi}{4}}{\sin\frac{5\pi}{4}}$

$$=\frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}$$

$$=-\frac{\sqrt{2}}{2}\times\left(-\frac{2}{\sqrt{2}}\right)$$

$$=1$$

i. $\cos 570^\circ=\cos(360^\circ+210^\circ)$

$$=\cos 210^\circ$$

$$=-\frac{\sqrt{3}}{2}$$

j. $\tan\left(-\frac{8\pi}{3}\right) = \tan\left(-2\pi - \frac{2\pi}{3}\right)$

$$= \tan\left(-\frac{2\pi}{3}\right)$$

$$= \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$$

$$= -\frac{\sqrt{3}}{2} \times \left(-\frac{2}{1}\right)$$

$$= \sqrt{3}$$

6. a. $\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

b. $\theta = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

c. $\theta = \frac{4\pi}{3}$ or $\frac{5\pi}{3}$

d. $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$

e. $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

f. $\theta = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

7. a. $\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = 60°$ or $120°$

b. $\sec\theta = -\sqrt{2}$

$\theta = 135°$ or $225°$

c. $\cos\theta = -1$

$\theta = 180°$

d. $\cot\theta = 0$

$\theta = 90°$ or $270°$

8. $\tan\theta$ is undefined when $\theta \in \left\{\frac{\pi}{2} + k\pi\right\}$, where k is any integer.

9. $\cot\frac{5\pi}{6} = \frac{\cos\frac{5\pi}{6}}{\sin\frac{5\pi}{6}}$

$$= \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$= -\frac{\sqrt{3}}{2} \times \frac{2}{1}$$

$$= -\sqrt{3}$$

$\cot\frac{11\pi}{6} = \frac{\cos\frac{11\pi}{6}}{\sin\frac{11\pi}{6}}$

$$= \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{2} \times \left(-\frac{2}{1}\right)$$

$$= -\sqrt{3}$$

$\cot\frac{2\pi}{3} = \frac{\cos\frac{2\pi}{3}}{\sin\frac{2\pi}{3}}$

$$= \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= -\frac{1}{2} \times \frac{2}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}} \text{ or } -\frac{\sqrt{3}}{3}$$

$\cot\frac{5\pi}{3} = \frac{\cos\frac{5\pi}{3}}{\sin\frac{5\pi}{3}}$

$$= \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2} \times \left(-\frac{2}{\sqrt{3}}\right)$$

$$= -\frac{1}{\sqrt{3}} \text{ or } -\frac{\sqrt{3}}{3}$$

10. Remember to change your calculator to the radian mode.

a. $\sin\left(-\frac{7\pi}{3}\right)$

7 +/− × π ÷ 3 = sin

-0.866025403

$\therefore \sin\left(-\frac{7\pi}{3}\right) = -0.866$

b. $\tan \frac{11\pi}{15}$

1 1 × π ÷ 1 5 = tan

-1.110612515

$\therefore \tan \frac{11\pi}{15} = -1.111$

Section 2: Follow-up Activities

Extra Help

1. **a.** False **b.** False
c. True **d.** False

2. **a.**
$$\begin{aligned} 125° &= \frac{\pi}{180°} \times 125° \\ &= \frac{25\pi}{36} \end{aligned}$$

$$\begin{aligned} \theta &= \frac{a}{r} \\ \frac{25}{36}\pi &= \frac{12}{r} \\ r &= \frac{12 \times 36}{25\pi} \\ &\doteq 5.5 \end{aligned}$$

b.
$$\begin{aligned} \theta &= \frac{a}{r} \\ &= \frac{28}{6} \\ &= 4\frac{2}{3} \\ &\doteq 4.7 \text{ rad} \end{aligned}$$

c.
$$\begin{aligned} \theta &= \frac{a}{r} \\ 2.5\pi &= \frac{a}{15} \\ a &= 15 \times 2.5\pi \\ &\doteq 118 \end{aligned}$$

3. **a.**
$$1 \text{ rad} = \frac{180°}{\pi}$$
$$\begin{aligned} \therefore \frac{\pi}{6} \text{ rad} &= \frac{180°}{\pi} \times \frac{\pi}{6} \\ &= 30° \end{aligned}$$

b.
$$\begin{aligned} \frac{7\pi \text{ rad}}{12} &= \frac{180°}{\pi} \times \frac{7\pi}{12} \\ &= 105° \end{aligned}$$

c.
$$\begin{aligned} -\frac{11\pi \text{ rad}}{15} &= \frac{180°}{\pi} \times \left(-\frac{11\pi}{15}\right) \\ &= 12° \times (-11) \\ &= -132° \end{aligned}$$

d. $-\dfrac{8\pi \text{ rad}}{3} = \dfrac{180°}{\pi} \times \left(-\dfrac{8\pi}{3}\right)$
$= -480°$

4. a. $1° = \dfrac{\pi}{180°}$

$\therefore\ 240° = \dfrac{\pi}{180°} \times 240°$
$= \dfrac{4\pi}{3}$

b. $-135° = \dfrac{\pi}{180°} \times (-135°)$
$= -\dfrac{3\pi}{4}$

c. $-540° = \dfrac{\pi}{180°} \times (-540°)$
$= -3\pi$

d. $330° = \dfrac{\pi}{180°} \times 330°$
$= \dfrac{11\pi}{6}$

5. a. 3 6 cos

0.809016994

3 6 sin

0.587785252

Therefore, the coordinates are
$(\cos 36°, \sin 36°) \doteq (0.809, 0.588)$.

b. 2 1 5 cos

-0.819152044

2 1 5 sin

-0.573576436

Therefore, the coordinates are
$(\cos 215°, \sin 215°) \doteq (-0.819, -0.574)$.

c. 8 0 +/− cos

0.173648177

8 0 +/− sin

-0.984807753

Therefore, the coordinates are
$(\cos(-80°), \sin(-80°)) \doteq (0.174, -0.985)$.

6. Remember to change your calculator to the radian mode.

a. 9 × π ÷ 5 = sin

-0.587785252

$\therefore \sin \frac{9\pi}{5} \doteq -0.5878$

b. 2 +/− × π ÷ 3 = cos

-0.5

$\therefore \cos\left(-\frac{2\pi}{3}\right) = -0.5000$

c. 5 +/− × π ÷ 6 = tan

0.577350269

$\therefore \tan\left(-\frac{5\pi}{6}\right) \doteq 0.5774$

Enrichment

1. $1 \text{ rev} = 2\pi \text{ rad}$

$$\begin{aligned} \therefore\ 120 \text{ rev} &= 120 \times 2\pi \text{ rad} \\ &= 240\pi \text{ rad} \end{aligned}$$

The rate is 240π rad/h.
Thus, its angular velocity is 240π rad/60 min $= 4\pi$ rad/min.

2. $$\begin{aligned} w &= \frac{\theta}{t} \\ 4.2 &= \frac{\theta}{0.6} \\ \theta &= 0.6 \times 4.2 \\ &= 2.52 \end{aligned}$$

$$\begin{aligned} \theta &= \frac{a}{r} \\ a &= \theta r \\ &= 2.52 \times 30 \\ &= 75.6 \end{aligned}$$

The length of the arc through which the pendulum swings is 75.6 cm.

3.

Central Angle (degrees)	Central Angle (radians)	Sector Area
30	$\frac{\pi}{6}$	$\frac{30}{360} \times$ area of unit circle $= \frac{\pi}{12}$
75	$\frac{5\pi}{12}$	$\frac{5\pi}{24}$
220	$\frac{11\pi}{9}$	$\frac{11\pi}{18}$
180	π	$\frac{\pi}{2}$

4.

Radius	Area of the Circle	Area of the Sector
1	π	$\frac{5\pi}{18}$
2	4π	$\frac{10\pi}{9}$
3	9π	$\frac{5\pi}{2}$
4	16π	$\frac{40\pi}{9}$

5. The area of a sector of a unit circle is given by $A = \frac{1}{2}\theta$, where θ is measured in radians.

The general formula for the area of a sector of any circle is $A = \frac{1}{2}r^2\theta$, where θ is measured in radians.

Section 3: Activity 1

1. Use the unit circle to find the value of θ.

a. $\theta = \frac{4\pi}{3}$ and $\frac{5\pi}{3}$

b. $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and $\frac{7\pi}{4}$

c. $\theta = 0, \pi$, and 2π

2. Use the unit circle to find the value of θ.

a. $\theta = \frac{2\pi}{3}$

b. $\theta = \frac{\pi}{4}$ and $\frac{3\pi}{4}$

c. $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$

3. Use the unit circle to find the value of θ.

a. $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$, and $\frac{4\pi}{3}$

b. $\theta = \frac{\pi}{3}$ and $\frac{2\pi}{3}$

c. $\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$

4. a. $2\cos x = -1$

$\cos x = -\frac{1}{2}$

$x = 120°$ and $x = 240°$

b. $5\cot x + 5 = 0$

$5\cot x = -5$

$\cot x = -1$

$x = 135°$ and $x = 315°$

c. $2\sin x = \sqrt{3}$

$\sin x = \frac{\sqrt{3}}{2}$

$x = 60°$ and $x = 120°$

5. a. $\cos(-x) = -\frac{1}{2}$

$\cos x = -\frac{1}{2}$ (since $\cos(-x) = \cos x$)

1 +/_ ÷ 2 = INV cos

120.

Cosine is negative in the second and third quadrants. Therefore, $x=-240°$ and $x=-120°$.

b. $\sqrt{3}\csc x=2$

$$\csc x=\frac{2}{\sqrt{3}}$$

$$\sin x=\frac{1}{\csc x}=\frac{1}{\frac{2}{\sqrt{3}}}=\frac{\sqrt{3}}{2}$$

3 √ ÷ 2 = INV sin

60.

Sine is positive in the second quadrant. Therefore, $x=-240°$.

c. $4\sin(-x)+1=0$

$$4\sin(-x)=-1$$

$$\sin(-x)=-\frac{1}{4}$$

$$-\sin x=-\frac{1}{4}$$

$$\sin x=\frac{1}{4}$$

1 ÷ 4 = INV sin

14.4775 12 19

$\sin(-x)=-\sin x$

Sine is positive in the second quadrant. Therefore, $x \doteq -194°$.

d. $-2\sin x-4+3=0$

$$-2\sin x-1=0$$

$$-2\sin x=1$$

$$\sin x=-\frac{1}{2}$$

1 ÷ 2 +/_ = INV sin

-30.

Sine is negative in the third quadrant. Therefore, $x=-150°$.

Section 3: Activity 2

1. a. $\sin^2\theta = \frac{3}{4}$

$\sin\theta = \pm\frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ and } \frac{5\pi}{3}$

b. $2\tan^2\theta - 2 = 0$

$2\tan^2\theta = 2$

$\tan^2\theta = 1$

$\tan\theta = \pm 1$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ and } \frac{7\pi}{4}$

c. $\cos^2\theta - \frac{1}{4} = 0$

$\cos^2\theta = \frac{1}{4}$

$\cos\theta = \pm\frac{1}{2}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \text{ and } \frac{5\pi}{3}$

d. $\sec^2\theta = 1$

$\frac{1}{\cos^2\theta} = 1$

$\cos^2\theta = 1$

$\cos\theta = \pm 1$

$\theta = 0, \pi, \text{ and } 2\pi$

2. a. $\sin\theta\cos\theta - \sin\theta = 0$

$\sin\theta(\cos\theta - 1) = 0$

$\therefore \sin\theta = 0$ or $\cos\theta - 1 = 0$

$\theta = 0, \pi, \text{ and } 2\pi$ $\qquad \cos\theta = 1$

$\theta = 0 \text{ and } 2\pi$

Since $\frac{\pi}{2} \le \theta \le 2\pi$, $\theta = \pi$ and 2π.

b. $\cot^2\theta = \cot\theta$

$\cot^2\theta - \cot\theta = 0$

$\cot\theta(\cot\theta - 1) = 0$

$\therefore \cot\theta = 0$ or $\cot\theta - 1 = 0$

$\theta = \frac{\pi}{2} \text{ and } \frac{3\pi}{2}$ $\qquad \cot\theta = 1$

$\theta = \frac{\pi}{4} \text{ and } \frac{5\pi}{4}$

Since $\frac{\pi}{2} \le \theta \le 2\pi$, $\theta = \frac{\pi}{2}, \frac{5\pi}{4}, \text{ and } \frac{3\pi}{2}$.

c. $\tan^2\theta - \tan\theta = 0$

$\tan\theta(\tan\theta - 1) = 0$

$\therefore\ \tan\theta = 0$ or $\tan\theta - 1 = 0$

$\theta = \pi \text{ and } 2\pi$ $\qquad \tan\theta = 1$

$\theta = \dfrac{5\pi}{4}$

Since $\frac{\pi}{2} \le \theta \le 2\pi$, $\theta = \pi$, $\frac{5\pi}{4}$, and 2π.

d. $\cos\theta\ \csc\theta - 2\cos\theta = 0$

$\cos\theta(\csc\theta - 2) = 0$

$\cos\theta = 0$ or $\csc\theta - 2 = 0$

$\theta = \dfrac{\pi}{2} \text{ and } \dfrac{3\pi}{2}$ $\qquad \csc\theta = 2$

$\dfrac{1}{\sin\theta} = 2$

$2\sin\theta = 1$

$\sin\theta = \dfrac{1}{2}$

$\theta = \dfrac{5\pi}{6}$

Therefore, $\theta = \frac{\pi}{2}$, $\frac{5\pi}{6}$, and $\frac{3\pi}{2}$.

3. a. $2\sin^2 x - 3\sin x + 1 = 0$

$(2\sin x - 1)(\sin x - 1) = 0$

$2\sin x - 1 = 0$ or $\sin x - 1 = 0$

$2\sin x = 1$ $\qquad \sin x = 1$

$\sin x = \dfrac{1}{2}$ $\qquad x = 90°$

$x = 30° \text{ and } 150°$

Therefore, $x = 30°, 90°, \text{ and } 150°$.

b. $\sin^2 x - \cos^2 x = 0$

$(\sin x - \cos x)(\sin x + \cos x) = 0$

$\sin x - \cos x = 0$

$\sin x = \cos x$

$\dfrac{\sin x}{\cos x} = \dfrac{\cos x}{\cos x}$

$\tan x = 1$

$x = 45° \text{ and } 225°$ or

$\sin x + \cos x = 0$

$\sin x = -\cos x$

$\dfrac{\sin x}{\cos x} = \dfrac{-\cos x}{\cos x}$

$\tan x = -1$

$x = 135° \text{ and } 315°$

Therefore, $x = 45°, 135°, 225°, \text{ and } 315°$.

4. a. $2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$

$2\sin\theta - 1 = 0$ or $\sin\theta + 1 = 0$

$\sin\theta = \frac{1}{2}$ $\quad$ $\sin\theta = -1$

$\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$ $\quad$ $\theta = \frac{3\pi}{2}$

Therefore, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, and $\frac{3\pi}{2}$.

b. $6\cos^2\theta + 5\cos\theta - 4 = 0$

$(2\cos\theta - 1)(3\cos\theta + 4) = 0$

$2\cos\theta - 1 = 0$ or $3\cos\theta + 4 = 0$

$\cos\theta = \frac{1}{2}$ $\quad$ $\cos\theta = \frac{-4}{3}$

$\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$

Since the maximum value of $\cos\theta$ is 1 and the minimum value is -1, there is no solution for $\cos\theta = \frac{-4}{3}$. Therefore, $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$.

5. a. $1 - \tan\theta = 0$ or $1 + \cos\theta = 0$

$\tan\theta = 1$ $\quad$ $\cos\theta = -1$

$\theta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$ $\quad$ $\theta = \pi$

Therefore, $\theta = \frac{\pi}{4}, \pi$, and $\frac{5\pi}{4}$.

b. $\sin^2\theta - 2\sin\theta\cos\theta = -\cos^2\theta$

$\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = 0$

Let $y = \sin\theta$ and $x = \cos\theta$.

$$y^2 - 2yx + x^2 = 0$$
$$(y-x)^2 = 0$$
$$y - x = 0$$
$$y = x$$

$\therefore \sin\theta = \cos\theta$

This only occurs when $\theta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$.

or $\frac{\sin\theta}{\cos\theta} = 1$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$ and $\frac{5\pi}{4}$

6. a. Convert the time to decimal hours on the 24-hour clock.

$t = 6{:}30$ A.M.

$= 06.50$ h

Substitute $t = 6.50$ h in

$$h = 3\cos 2\pi\frac{(t-4.5)}{12.4} + 5$$
$$= 3\cos 2\pi\frac{(6.5-4.5)}{12.4} + 5$$
$$= 3\cos 2\pi\left(\frac{2}{12.4}\right) + 5$$
$$\doteq 8.0402$$

At 6:30 A.M., the depth of the water is 8.0 m.

b. Substitute $h = 7.5$ m in

$$h = 3\cos 2\pi\frac{(t-4.5)}{12.4} + 5$$

$$7.5 = 3\cos 2\pi\frac{(t-4.5)}{12.4} + 5$$

$$2.5 = 3\cos 2\pi\frac{(t-4.5)}{12.4}$$

$$\cos 2\pi\frac{(t-4.5)}{12.4} = \frac{2.5}{3}$$

Find inverse cos of $\frac{2.5}{3}$.

$$2\pi\frac{(t-4.5)}{12.4} = 0.585\,686$$

$$t - 4.5 = \frac{(0.585\,686)(12.4)}{2\pi}$$

$$t = 4.5 + 1.155\,864$$

$$\doteq 5.655\,864$$

To convert t to the time of the day, multiply the decimal part by 60 since there are 60 min in an hour.

$$5.655\,864 = 5 + 0.655\,864 \times 60$$

$$= 5 + 39.351\,84$$

$$\doteq 5.39$$

The depth of the water is 7.5 m at 5:39 A.M.

7. Substitute $t = 8$ in

$$h = 15\cos\frac{2\pi t}{5} + 24$$

$$= 15\cos\frac{2\pi(8)}{5} + 24$$

$$\doteq 11.8647$$

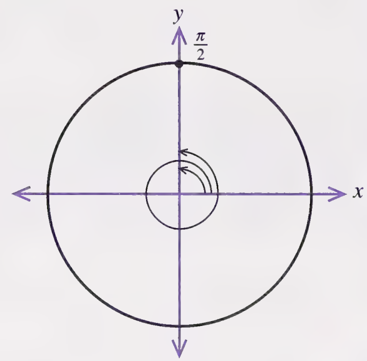
After 8 s, the height of the pedal is 11.9 m.

Section 3: Activity 3

1. a. $\sin 2\theta = 1$

$$2\theta = \frac{\pi}{2} \text{ and } \frac{5\pi}{2}$$

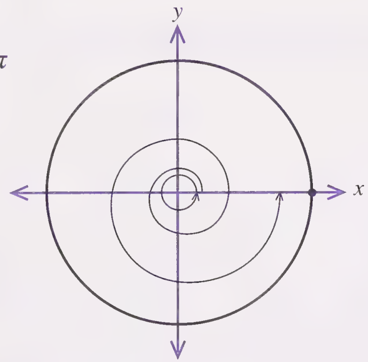
$$\theta = \frac{\pi}{4} \text{ and } \frac{5\pi}{4}$$



b. $\cos 2\theta = 1$

$2\theta = 0,\ 2\pi,\ \text{and } 4\pi$

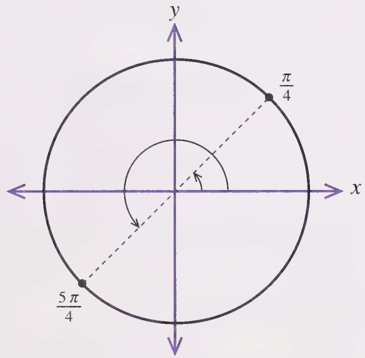
$\theta = 0,\ \pi,\ \text{and } 2\pi$



c. $\tan 2\theta = 1$

$$2\theta = \frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{\pi}{4}+2\pi,\ \text{and } \frac{5\pi}{4}+2\pi$$

$$\theta = \frac{\pi}{8},\ \frac{5\pi}{8},\ \frac{9\pi}{8},\ \text{and } \frac{13\pi}{8}$$



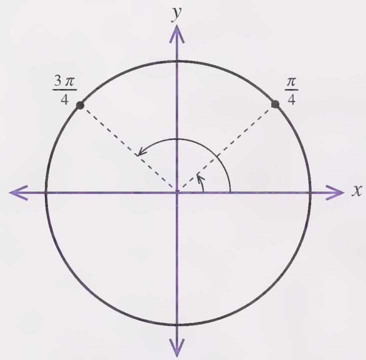
d. $\sqrt{2}\sin 2\theta = 1$

$$\sin 2\theta = \frac{1}{\sqrt{2}}$$

$$\sin 2\theta = \frac{\sqrt{2}}{2}$$

$$2\theta = \frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{\pi}{4}+2\pi,\ \text{and } \frac{3\pi}{4}+2\pi$$

$$\theta = \frac{\pi}{8},\ \frac{3\pi}{8},\ \frac{9\pi}{8},\ \text{and } \frac{11\pi}{8}$$



2. **a.** $4\sin^2 2\theta = 1$

$$\sin^2 2\theta = \frac{1}{4}$$

$$\sin 2\theta = \pm\frac{1}{2}$$

$$2\theta = 30°,\ 150°,\ 210°,\ 330°,\ 390°,\ 510°,\ 570°,\ 690°$$

$$\theta = 15°,\ 75°,\ 105°,\ 165°,\ 195°,\ 255°,\ 285°,\ 345°$$

b. $2\sin^2 2\theta - 1 = -\sin 2\theta$

Let $\alpha = 2\theta$.

$$2\sin^2 \alpha - 1 = -\sin \alpha$$
$$2\sin^2 \alpha + \sin \alpha - 1 = 0$$
$$(2\sin \alpha - 1)(\sin \alpha + 1) = 0$$
$$2\sin \alpha - 1 = 0 \quad \text{or} \quad \sin \alpha + 1 = 0$$
$$\sin \alpha = \frac{1}{2} \qquad \sin \alpha = -1$$

$\alpha = 30°, 150°, 270°, 390°, 510°,$ and $630°$

$2\theta = 30°, 150°, 270°, 390°, 510°,$ and $630°$

$\theta = 15°, 75°, 135°, 195°, 255°,$ and $315°$

c. $\tan 3\theta = \sqrt{3}$

Let $\alpha = 3\theta$. Thus, $\tan \alpha = \sqrt{3}$, where $0 \le \alpha \le 1080°$.

Notice the domain for α is three times the domain for θ.

$\alpha = 60°, 240°, 420°, 600°, 780°,$ and $960°$

$3\theta = 60°, 240°, 420°, 600°, 780°,$ and $960°$

$\theta = 20°, 80°, 140°, 200°, 260°,$ and $320°$

d. $\sin\left(\frac{1}{2}\theta\right) = -\frac{1}{2}$

Let $\alpha = \frac{1}{2}\theta$. Thus, $\sin \alpha = -\frac{1}{2}$, where $0 \le \alpha \le 180°$.

Notice the domain for α is now $\frac{1}{2}$ the domain for θ. Since there is no value of α from 0 to 180°, where sine is $-\frac{1}{2}$, then there is no solution for this equation for the given domain.

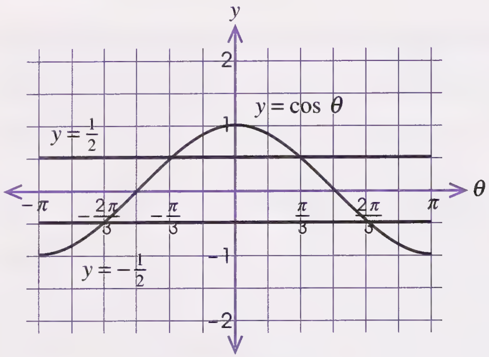
3.
$$d = \frac{1}{10}v_0^2 \sin 2\theta$$
$$120 = \frac{1}{10}(40)^2 \sin 2\theta$$
$$120 = 160 \sin 2\theta$$
$$\sin 2\theta = \frac{120}{160}$$
$$\sin 2\theta = 0.75$$
$$2\theta = 48.6° \quad \text{or} \quad 131.4°$$
$$\theta = 24.3° \quad \text{or} \quad 65.7°$$

Section 3: Activity 4

1. a.

θ	$-\pi$	$-\frac{2\pi}{3}$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
$y = \cos \theta$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1

Graph $y = \cos \theta$, $y = \frac{1}{2}$, and $y = -\frac{1}{2}$. The intercepts of the graphs $y = \cos \theta$ and $y = \frac{1}{2}$, and the intercepts of the graphs of $y = \cos \theta$ and $y = -\frac{1}{2}$ are the roots of the equation in the interval $-\pi \le \theta \le \pi$.



The roots are $-\frac{2\pi}{3}$, $-\frac{\pi}{3}$, $\frac{\pi}{3}$, and $\frac{2\pi}{3}$.

b. $4\cos^2\theta - 1 = 0$

$$\cos^2\theta = \frac{1}{4}$$

$$\cos\theta = \pm\frac{1}{2}$$

Referring to the unit circle in the interval $-\pi \le \theta \le \pi$, $\cos\theta = \pm\frac{1}{2}$ at $\theta = -\frac{2\pi}{3}$, $-\frac{\pi}{3}$, $\frac{\pi}{3}$, and $\frac{2\pi}{3}$.

c. The graphical solutions and algebraic solutions are the same.

2. a. $\sin x - \cos x = 0$

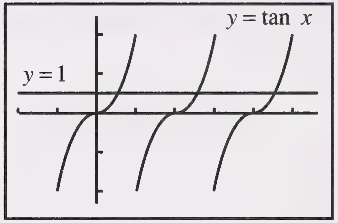
$$\sin x = \cos x$$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

Graph $y = \tan x$ and $y = 1$.

Range
Xmin : –90
max : 360
scl : 90
Ymin : –6
max : 6
scl : 2



The solutions are approximately 43.5° and 223.5°.

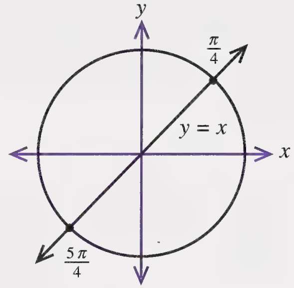
If you want more accurate values, use the Zoom function. (Refer to your manual for instructions.) The actual values are 45° and 225°.

b. The general form for 45° is $45° + 360n°$.
The general form for 225° is $225° + 360n°$.

Section 3: Follow-up Activities

Extra Help

1. By definition, $\tan\theta = \frac{y}{x}$.



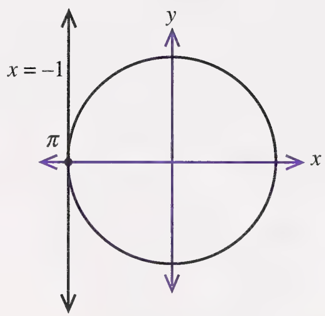
In this case, $\frac{y}{x} = 1$

$$y = x$$

This occurs when $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$.
(These are the only points on the unit circle that have the same x- and y-coordinates.)

The solution set is $\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$.

2. By definition, $\cos\theta = x$.

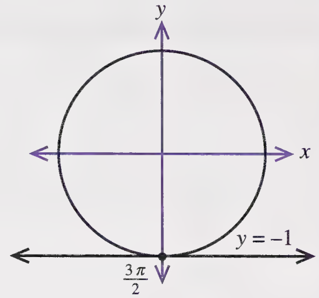


In this case, $x = -1$.

This occurs when $\theta = \pi$.

The solution set is $\{\pi\}$.

3. Let y be $\sin\theta$.

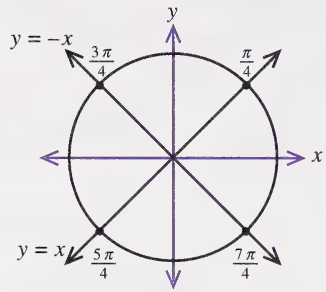


$$\begin{aligned} y^3 &= -1 \\ y &= \sqrt[3]{-1} \\ &= -1 \end{aligned}$$

This occurs when $\theta = \frac{3\pi}{2}$.

The solution set is $\left\{\frac{3\pi}{2}\right\}$.

4. $$\begin{aligned} \left(\frac{y}{x}\right)^2 &= 1 \\ \frac{y^2}{x^2} &= 1 \\ y^2 &= x^2 \\ y &= \pm x \end{aligned}$$

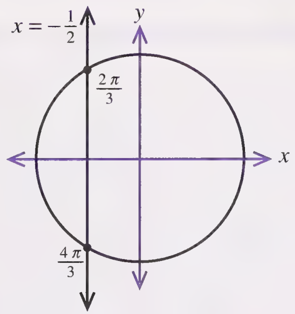


The solution set is $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$.

5. $$\begin{aligned} 5 + 5\cos\theta &= 3\cos\theta + 4 \\ 5 + 5x &= 3x + 4 \\ 2x &= -1 \\ x &= -\frac{1}{2} \end{aligned}$$

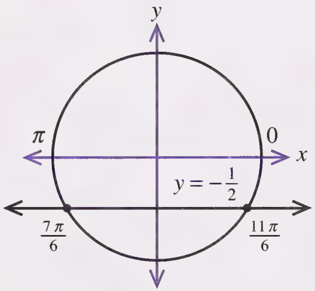
Note that the vertical line defined by $x=-\frac{1}{2}$ cuts the unit circle at $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

The solution set is $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.



6. $2\sin^2\theta+\sin\theta=0$

$$2y^2+y=0$$
$$y(2y+1)=0$$



$2y+1=0$ or $y=0$

$2y=-1$

$y=-\dfrac{1}{2}$

The graph of $y=-\frac{1}{2}$ cuts the unit circle at $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

That is, $\theta=\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

The graph of $y=0$ cuts the unit circle at 0 and π.

That is, $\theta=0$ or π.

The solution set is $\left\{0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.

7. $\tan^2\theta-\tan\theta=0$

$$\left(\frac{y}{x}\right)^2-\frac{y}{x}=0$$
$$\frac{y}{x}\left(\frac{y}{x}-1\right)=0$$

$\dfrac{y}{x}=0$ or $\dfrac{y}{x}-1=0$

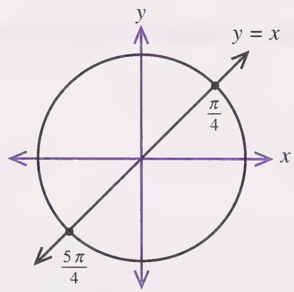
$y=0$

$\theta=0$ and π

$\dfrac{y}{x}=1$

$y=x$

$\theta=\dfrac{\pi}{4}$ and $\dfrac{5\pi}{4}$



The solution set is $\left\{0, \pi, \frac{\pi}{4}, \frac{5\pi}{4}\right\}$.

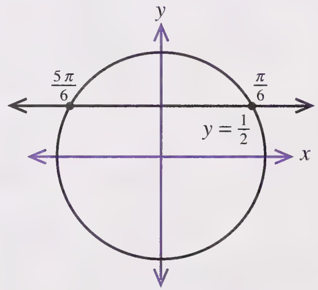
8. $2\sin^2\theta+7\sin\theta-4=0$

$$2y^2+7y-4=0$$
$$(2y-1)(y+4)=0$$

$2y-1=0$ or $y+4=0$

$2y=1$ $\qquad y=-4$

$y=\dfrac{1}{2}$



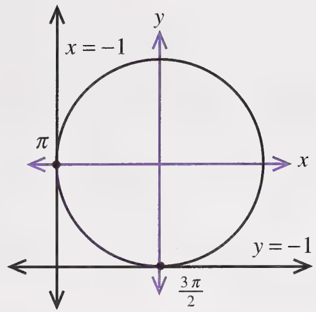
Since $y=-4$ never intersects the unit circle, there is no solution. Therefore, $\theta=\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

The solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$.

9. $(\sin\theta)(\cos\theta)+\sin\theta+\cos\theta+1=0$

$$\begin{aligned}(y)(x)+y+x+1&=0\\ y(x+1)+1(x+1)&=0\\ (y+1)(x+1)&=0\end{aligned}$$

$$\begin{array}{lcl} x+1=0 & \text{and} & y+1=0 \\ x=-1 & & y=-1 \\ \theta=\pi & & \theta=\dfrac{3\pi}{2} \end{array}$$



The solution set is $\left\{\pi, \frac{3\pi}{2}\right\}$.

Enrichment

1. $-2\cos^2\theta+2\cos\theta+3=0$, where $0 \le \theta \le 360°$

$$\cos\theta=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\cos\theta=\frac{-2\pm\sqrt{(2)^2-4(-2)(3)}}{2(-2)}$$

$$\cos\theta=\frac{-2\pm\sqrt{4+24}}{-4}$$

$$\cos\theta=\frac{-2\pm\sqrt{28}}{-4}$$

$\cos\theta \doteq -0.822\,876$ or $\cos\theta \doteq 1.822\,876$

$\theta \doteq 145.38°$

Another root of θ is $180°+145.38°=325.37°$.

No real values satisfy $\cos\theta \doteq 1.822\,876$. Therefore, $\theta \doteq 145.38°$ and $325.7°$.

2. $3\tan^2 x + \tan x - 1 = 0$, where $0 \le x \le 360°$

$$\tan x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\tan x = \frac{-1 \pm \sqrt{(1)^2 - 4(3)(-1)}}{2(3)}$$

$$\tan x = \frac{-1 \pm \sqrt{1+12}}{6}$$

$$\tan x = \frac{-1 \pm \sqrt{13}}{6}$$

$\tan x \doteq 0.434\ 259$ or $\tan x \doteq -0.767\ 592$

$x \doteq 23.47°$ $\qquad x \doteq -37.51°$

Another root of x is $180° + 23.47° = 203.47°$.

Since $x \doteq -37.51°$ does not lie in the interval, you must determine its coterminal angle.

$$\begin{aligned} x &= 360° - 37.51° \\ &= 322.49° \end{aligned}$$

Another root of x is $322.49° - 180° = 142.49°$.

Therefore, $x \doteq 23.47°, 142.49°, 203.47°,$ and $322.49°$.

3. $2\sin^2\theta - 4\sin\theta - 1 = 0$, where $180° \le \theta \le 360°$

$$\sin\theta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\sin\theta = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(-1)}}{2(2)}$$

$$\sin\theta = \frac{4 \pm \sqrt{16+8}}{4}$$

$$\sin\theta = \frac{4 \pm \sqrt{24}}{4}$$

$\sin\theta \doteq 2.224\ 745$ or $\sin\theta \doteq -0.224\ 745$

$\theta \doteq -12.99°$

No real roots satisfy $\sin\theta \doteq 2.224\ 745$.

Since $\theta \doteq -12.99°$ does not lie in the interval, you must determine its coterminal angle.

$$\begin{aligned} \theta &\doteq 360° - 12.99° \\ &\doteq 347.01° \end{aligned}$$

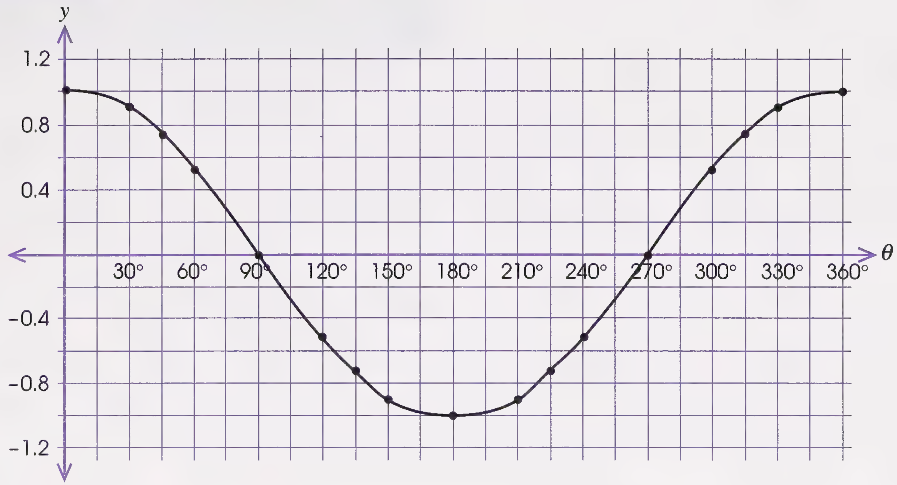
Another root of θ is $180° + 12.99° = 192.99°$.

Therefore, $\theta \doteq 192.99°$ and $347.01°$.

Section 4: Activity 1

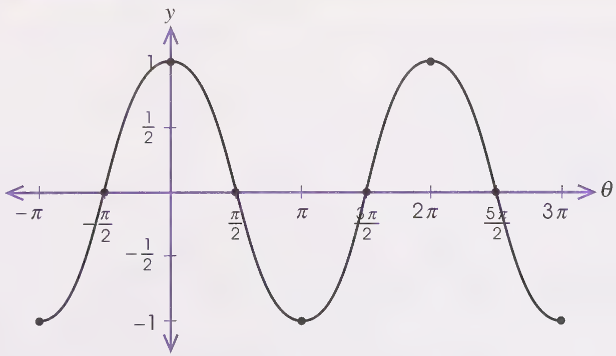
1.

θ	radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
	degrees	0	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
$y = \cos\ \theta$	exact	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
	approximate	1.0	0.9	0.7	0.5	0	-0.5	-0.7	-0.9	-1	-0.9	-0.7	-0.5	0	0.5	0.7	0.9	1.0



2.

θ	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π
$y=\cos\theta$	-1	0	1	0	-1	0	1	0	-1



3. **a.** The domain is $\{\theta | \theta \in R\}$. **b.** The range is $\{y | -1 \le y \le 1\}$.

   c. The period is 2π. **d.** The amplitude is 1.

4.

Function	Domain	Range	Period
$y=\sin\theta$	$\{\theta \vert \theta \in R\}$	$\{y \vert -1 \le y \le 1\}$	2π
$y=\cos\theta$	$\{\theta \vert \theta \in R\}$	$\{y \vert -1 \le y \le 1\}$	2π
$y=\tan\theta$	$\{\theta \vert \theta \ne \frac{\pi}{2}+n\pi,\ n \in I\}$	$\{y \vert y \in R\}$	π

Section 4: Activity 2

1. **a.** $|a|=|0.5|$ $=0.5$ $\quad p=\frac{2\pi}{b}$ $=\frac{2\pi}{4}$ $=\frac{\pi}{2}$

   b. $|a|=|1|$ $=1$ $\quad p=\frac{2\pi}{b}$ $=\frac{2\pi}{\frac{1}{5}}$ $=10\pi$

   c. $|a|=|-4|$ $=4$ $\quad p=\frac{2\pi}{b}$ $=\frac{2\pi}{\frac{2}{3}}$ $=3\pi$

   d. $|a|=|3|$ $=3$ $\quad p=\frac{2\pi}{b}$ $=\frac{2\pi}{4}$ $=\frac{\pi}{2}$

   e. $|a|=|-1|$ $=1$ $\quad p=\frac{2\pi}{|b|}$ $=\frac{2\pi}{|-2|}$ $=\pi$

   f. $y=2\cos(2\theta+\pi)$ $=2\cos\left[2\left(\theta+\frac{\pi}{2}\right)\right]$

   $|a|=|2|$ $=2$ $\quad p=\frac{2\pi}{b}$ $=\frac{2\pi}{2}$ $=\pi$

2. a. $y = \frac{1}{2}\sin\theta - 2.5$

$= \frac{1}{2}\sin(\theta + 0) - 2.5$

The phase shift is 0, and the vertical translation is -2.5.

b. $y = \cos\left(2\theta - \frac{\pi}{6}\right)$

$= \cos\left[2\left(\theta - \frac{\pi}{12}\right)\right]$

The phase shift is $\frac{\pi}{12}$ to the right, and the vertical translation is 0.

c. $f(\theta) = 3\sin(4\theta + 3\pi) + 2$

$= 3\sin\left[4\left(\theta + \frac{3\pi}{4}\right)\right] + 2$

The phase shift is $\frac{3\pi}{4}$ to the left, and the vertical translation is 2.

d. The phase shift is $\frac{\pi}{4}$ to the left; the vertical translation is 7.

e. $f(\theta) = 2\cos(3\theta - \pi) + 3$

$= 2\cos\left[3\left(\theta - \frac{\pi}{3}\right)\right] + 3$

The phase shift is $\frac{\pi}{3}$ to the right, and the vertical translation is 3.

f. The phase shift is $\frac{\pi}{4}$ to the left; the vertical translation is 5.

3. a. $y = 3\cos(2\theta - \pi) - 3.5$

$= 3\cos\left[2\left(\theta - \frac{\pi}{2}\right)\right] - 3.5$

$|a| = |3|$
$= 3$

$p = \frac{2\pi}{b}$
$= \frac{2\pi}{2}$
$= \pi$

The amplitude is 3; the period is π; the phase shift is $\frac{\pi}{2}$ to the right; and the vertical translation is -3.5.

b. $f(x) = -2\sin\left(\frac{1}{2}\theta + \frac{\pi}{2}\right) + 2$

$= -2\sin\left[\frac{1}{2}(\theta + \pi)\right] + 2$

$|a| = |2|$
$= 2$

$p = \frac{2\pi}{b}$
$= \frac{2\pi}{\frac{1}{2}}$
$= 4\pi$

The amplitude is 2; the period is 4π; the phase shift is π to the left; and the vertical translation is 2.

4. a. The period is 365 days.

b. $a = \frac{M - m}{2}$

$= \frac{17 - 7.5}{2}$

$= 4.75$

5. a. Since there is no phase shift and vertical translation, the form is $y = a \sin b\theta$.

$a = 2$

$$\begin{aligned} p &= \frac{2\pi}{b} \\ -\pi &= \frac{2\pi}{b} \\ b &= \frac{2\cancel{\pi}}{-\cancel{\pi}} \\ &= -2 \end{aligned}$$

Therefore, $y = a \sin b\theta$ becomes $y = 2 \sin(-2\theta)$.

b. Since there is no vertical translation, the form is $y = a \sin[b(\theta + c)]$.

$a = -3$

$c = \frac{\pi}{2}$

$$\begin{aligned} p &= \frac{2\pi}{b} \\ \frac{2\pi}{3} &= \frac{2\pi}{b} \\ b &= \frac{3(\cancel{2\pi})}{\cancel{2\pi}} \\ &= 3 \end{aligned}$$

Therefore, $y = a \sin[b(\theta + c)]$ becomes $y = -3 \sin\left[3\left(\theta + \frac{\pi}{2}\right)\right]$.

6. a. Since there is no vertical translation, $y = a \cos[b(\theta + c)]$.

$a = 1$

$c = -\frac{\pi}{3}$

$$\begin{aligned} p &= \frac{2\pi}{b} \\ 2\pi &= \frac{2\pi}{b} \\ b &= \frac{\cancel{2\pi}}{\cancel{2\pi}} \\ &= 1 \end{aligned}$$

Therefore, $y = a \cos[b(\theta + c)]$ becomes $y = \cos\left(\theta - \frac{\pi}{3}\right)$.

b. The form is $y = a \cos[b(\theta + c)] + d$.

$a = -5$

$c = \frac{\pi}{4}$

$d = -2$

$$\begin{aligned} p &= \frac{2\pi}{b} \\ \frac{\pi}{3} &= \frac{2\pi}{b} \\ b &= \frac{3(2\cancel{\pi})}{\cancel{\pi}} \\ &= 6 \end{aligned}$$

Therefore, $y = a \cos[b(\theta + c)] + d$ becomes $y = -5 \cos\left[6\left(\theta + \frac{\pi}{4}\right)\right] - 2$.

Section 4: Activity 3

1. The amplitude is a. It is a vertical stretch of the graph $y = \cos\ \theta$. The maximum and minimum values of $y = a\cos\ \theta$ occur for the same values of θ as in $y = \cos\ \theta$. The magnification is caused by a factor a.

2. **Graph A**

a. $$\begin{aligned}|a| &= \frac{M-m}{2}\\ &= \frac{6-(-6)}{2}\\ &= 6\end{aligned}$$

The amplitude is 6.

b. The phase shift is $\frac{\pi}{4}$ units to the left for the sine function, or $\frac{\pi}{4}$ units to the right for the cosine function.

c. The period is 2π.

d. The maximum value is 6, and the minimum is -6.

e. $$\begin{aligned}d &= \frac{M+m}{2}\\ &= \frac{6+(-6)}{2}\\ &= 0\end{aligned}$$

There is no vertical translation.

f. The range is $-6 \le y \le 6$.

Graph B

a. $$\begin{aligned}|a| &= \frac{M-m}{2}\\ &= \frac{15-5}{2}\\ &= 5\end{aligned}$$

The amplitude is 5.

b. The phase shift is $\frac{\pi}{2}$ units to the left for the sine function, or 0 for the cosine function.

c. The period is 2π.

d. The maximum value is 15, and the minimum value is 5.

e. $$\begin{aligned}d &= \frac{M+m}{2}\\ &= \frac{15+5}{2}\\ &= 10\end{aligned}$$

The vertical translation is 10.

f. The range is $5 \le y \le 15$.

Graph C

a. $$\begin{aligned}|a| &= \frac{M-m}{2}\\ &= \frac{3-(-1)}{2}\\ &= 2\end{aligned}$$

The amplitude is 2.

b. The phase shift is meaningless.

c. The period is 8.

d. The maximum value is 3, and the minimum value is -1.

e. $$\begin{aligned}d &= \frac{M+m}{2}\\ &= \frac{3+(-1)}{2}\\ &= 1\end{aligned}$$

The vertical translation is 1.

f. The range is $-1 \le y \le 3$.

3. a. Graph A

$$\begin{aligned}p &= \frac{2\pi}{b}\\ 2\pi &= \frac{2\pi}{b}\\ b &= 1\end{aligned}$$

$$\begin{aligned}\therefore\ y &= a \sin\left[b(\theta + c)\right] + d\\ &= 6 \sin\left(\theta + \frac{\pi}{4}\right) \text{ or } 6 \cos\left(\theta - \frac{\pi}{4}\right)\end{aligned}$$

Graph B

$$\begin{aligned}p &= \frac{2\pi}{b}\\ 2\pi &= \frac{2\pi}{b}\\ b &= 1\end{aligned}$$

$$\begin{aligned}\therefore\ y &= a \sin\left[b(\theta + c)\right] + d\\ &= 5 \sin\left(\theta + \frac{\pi}{2}\right) + 10 \text{ or } 5 \cos \theta + 10\end{aligned}$$

b. $p = \dfrac{2\pi}{b}$

$8 = \dfrac{2\pi}{b}$

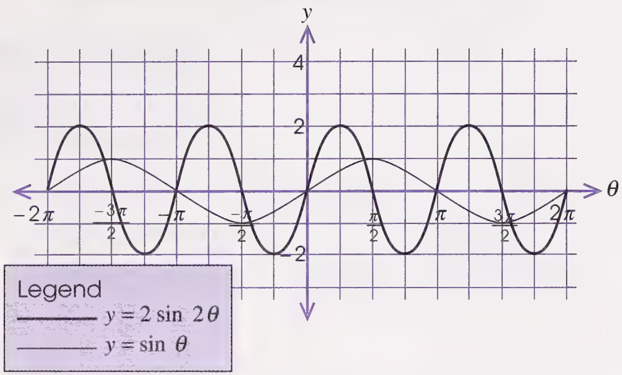
$b = \dfrac{\pi}{4}$

$y = a \sin\left[b(t+c)\right] + d$

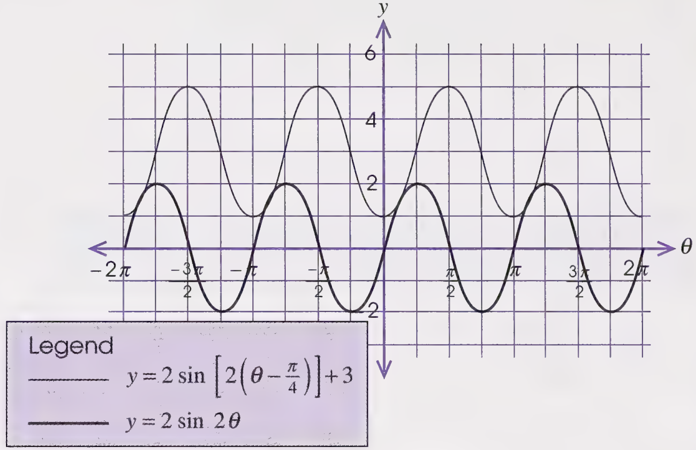
$= 2 \sin \dfrac{\pi}{4} t + 1$

4. The phase shift is c. If c is negative, the graph is shifted c units to the right. If c is positive, the graph is shifted c units to the left.

5. Draw the graphs of $y = \sin\ \theta$ and $y = 2 \sin\ 2\theta$ on the same axes. The period for $y = 2 \sin\ 2\theta$ is $\frac{2\pi}{2} = \pi$, and the amplitude is 2.



Move the graph $y = 2 \sin\ 2\theta$ three units up and $\frac{\pi}{4}$ units to the right. The vertical displacement is 3 and the phase shift is $\frac{\pi}{4}$ units.



The range is $1 \le y \le 5$.

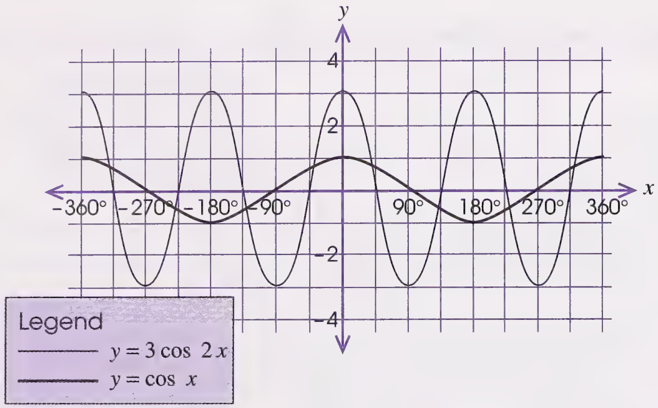
6. $y = 3 \cos\ (2x - 120°) - 1$

$= 3 \cos\left[2(x - 60°)\right] - 1$

The phase shift is 60° to the right, and the period is $\frac{360°}{2} = 180°$.

Draw the graphs of $y = \cos\ x$ and $y = 3 \cos\ 2x$ on the same axes.

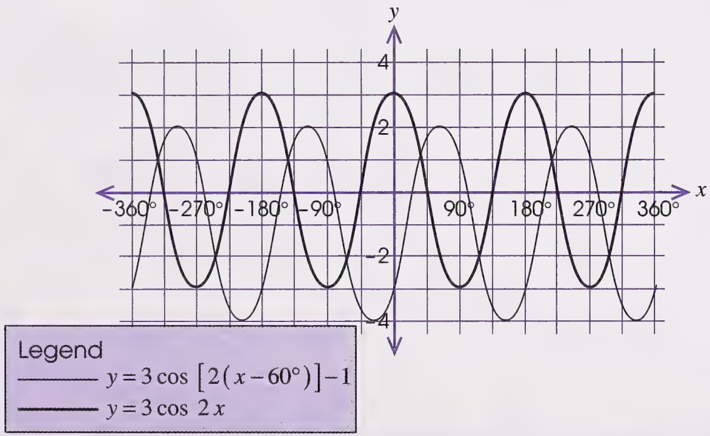
Note: x is substituted for θ since degrees are substituted for π.



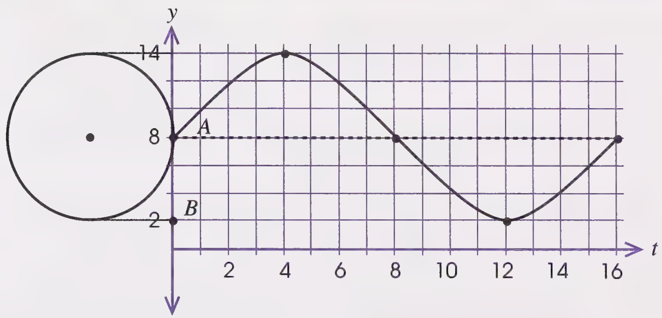
The phase shift is 60° to the right since

$y = 3\cos\left[2(x - 60°)\right] - 1.$

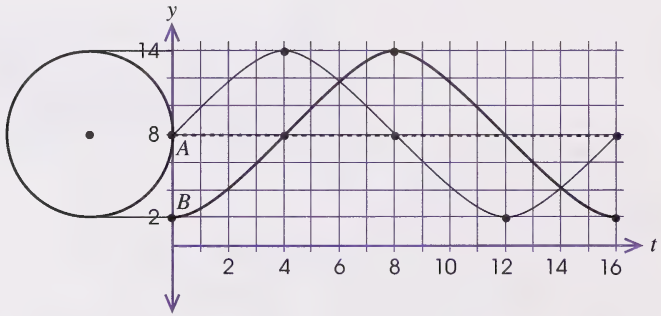
Thus, move the graph of $y = 3\cos 2x$ one unit down and 60° to the right.



7. a. Draw the basic sine curve at point A (initial point).



Now, draw the curve with initial point B, the lowest point on the Ferris wheel.



b. $y = a \sin\left[b(t+c)\right] + d$ is the general sine function.

From the graph, you can find the following:

$$\begin{aligned} |a| &= \frac{M-m}{2} \\ &= \frac{14-2}{2} \\ &= 6 \end{aligned} \qquad \begin{aligned} p &= \frac{2\pi}{b} \\ 16 &= \frac{2\pi}{b} \\ b &= \frac{\pi}{8} \end{aligned}$$

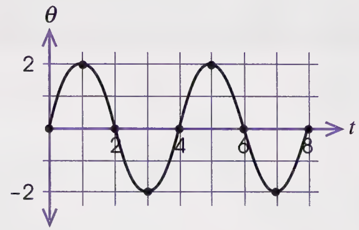
The phase shift is 4 units to the right; therefore, $c = -4$.

$$\begin{aligned} \text{vertical translation} &= \frac{M+m}{2} \\ &= \frac{14+2}{2} \\ &= 8 \end{aligned}$$

Therefore, the equation of the graph is
$y = 6 \sin\left[\frac{\pi}{8}(t-4)\right] + 8.$

8. a. $\theta = 2 \sin\left(\frac{1}{2}\pi t\right)$

t	0	1	2	3	4	5	6	7	8
θ	0	2	0	−2	0	2	0	−2	0



b. $\theta = 0$ for all positive even integers of t or $t = 0$

9.

Parameter	Description	Effects on the Graph of $y = \cos\ \theta$
$\frac{1}{2}$	amplitude	changes the distance between the maximum and minimum values from 2 to 1, or, the distance from the midline to the maximum and minimum decreases from 1 to $\frac{1}{2}$
$\frac{\pi}{4}$	horizontal translation	moves the graph of $y = \cos\ \theta$ to the left $\frac{\pi}{4}$ units
-3	vertical translation	moves the midline of $y = \frac{1}{2}\cos\left(\theta + \frac{\pi}{4}\right)$ down three units

10.

Answer	Response	Reason
amplitude $a = 12$	incorrect	The student forgot to take half the difference between maximum and minimum values. $\frac{M-m}{2} = \frac{4+8}{2} = 6$
period $p = \frac{4}{5}\pi$	incorrect	The student calculated incorrectly. The period is π.
range $\theta \in R$	incorrect	The student confused domain and range. The range is $-8 \le y \le 4$.

11.

Parameter	Description	Effects on the Graph of $y = \sin 2\theta$
2	amplitude	changes the distance between the maximum and minimum values from 2 to 4, or the distance from the midline to the maximum or minimum increases from 1 to 2
$\frac{\pi}{4}$	horizontal translation	moves the graph of $y = \sin 2\theta$ to the left $\frac{\pi}{4}$ units
-4	vertical translation	moves the midline of $y = 2\sin\left(2\theta + \frac{\pi}{2}\right)$ down four units

Section 4: Follow-up Activities

Extra Help

1. A vertical asymptote is a line that a graph approaches but never touches.

   The period is the horizontal distance from any point to the next point where the cycle begins again.

2. In the ordered pair (x, y), the domain is the set of all x-values and the range is the set of all y-values.

3. a. The period is $\frac{2\pi}{|b|}$.

   b. The phase shift is c. If c is positive, the phase shift is to the left. If c is negative, the phase shift is to the right.

   c. The vertical translation is 0.

   d. The amplitude is $|a|$.

   e. Range: $\{y \mid -a \le y \le a\}$

4. a. $|a| = |2|$

   $= 2$

   The amplitude is 2.

   b. $p = \frac{2\pi}{\frac{1}{3}}$

   $= 6\pi$

   The period is 6π.

   c. The phase shift is 0.

5. a. $|a| = \frac{M-m}{2}$

   $= \frac{2+2}{2}$

   $= 2$

   $y = \cos\theta$ is stretched by a factor of 2. Therefore,

   $y = 2\cos\theta$.

   b. $|a| = \frac{M-m}{2}$

   $= \frac{1+1}{2}$

   $= 1$

   The amplitude is 1, and the period is 4π.

   $p = \frac{2\pi}{b}$

   $4\pi = \frac{2\pi}{b}$

   $b = \frac{1}{2}$

   $\therefore\ y = \cos\frac{1}{2}\theta$

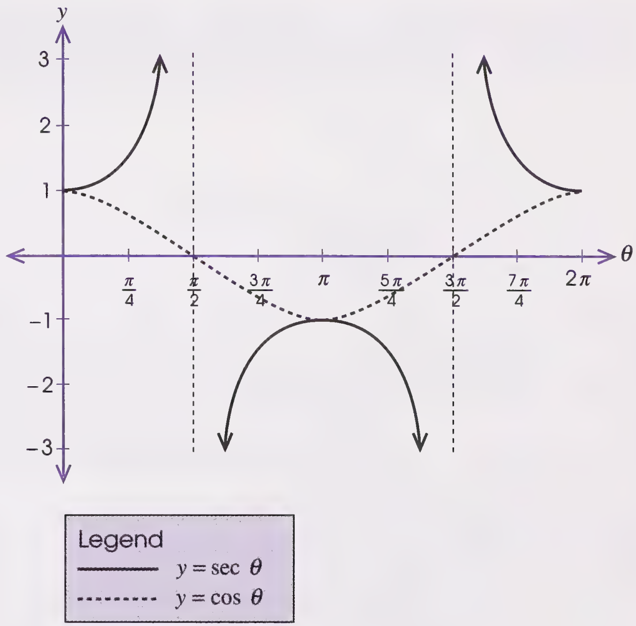
6. a. The maximum value is 6.
   b. The period is 20.
   c. Height is a minimum when $t = 15$ h.
   d. The t-intercepts are $t = 0$ h, $t = 10$ h, and $t = 20$ h.

Enrichment

1. a.

θ	$\cos\theta$	$\sec\theta$
0	1	1
$0<\theta<\frac{\pi}{2}$	positive and decreasing	positive and increasing
$\frac{\pi}{2}$	0	undefined
$\frac{\pi}{2}<\theta<\pi$	negative and decreasing	negative and increasing
π	−1	−1
$\pi<\theta<\frac{3\pi}{2}$	negative and increasing	negative and decreasing
$\frac{3\pi}{2}$	0	undefined
$\frac{3\pi}{2}<\theta<2\pi$	positive and increasing	positive and decreasing
2π	1	1

b.

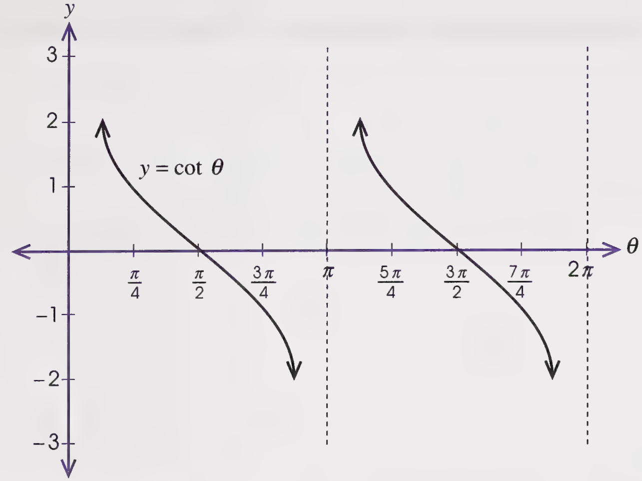


2. The domain is $\left\{\theta \middle| \theta \neq \frac{\pi}{2}+n\pi,\ n\in I\right\}$.

The range is $\left\{y \middle| y\leq -1 \text{ or } y\geq 1\right\}$.

The period is 2π.

3.



4. The domain is $\{\theta \mid \theta \neq n\pi,\ n \in I\}$.

The range is $\{y \mid y \in R\}$.

The period is π rad.

Section 5: Activity 1

1. a.

1 2 0 tan

-1.732050808

1 2 0 sin ÷ 1 2 0 cos =

-1.732050808

$$\therefore\ \tan\theta = \frac{\sin\theta}{\cos\theta}$$

b.

1 2 0 tan $\frac{1}{x}$

-0.577350269

1 2 0 cos ÷ 1 2 0 sin =

-0.577350269

$$\therefore\ \cot\theta = \frac{\cos\theta}{\sin\theta}$$

2. a.

LS	RS
$\sin 60°$	$\dfrac{1}{\csc 60°}$
$=\dfrac{\sqrt{3}}{2}$	$=\dfrac{1}{\frac{2}{\sqrt{3}}}$
	$=\dfrac{\sqrt{3}}{2}$

LS = RS

$$\therefore\ \sin\theta = \frac{1}{\csc\theta}$$

b.

LS	RS
$\sec 60°$	$\dfrac{1}{\cos 60°}$
$=\dfrac{2}{1}$	$=\dfrac{1}{\frac{1}{2}}$
$=2$	$=2$

LS = RS

$$\therefore\ \sec\theta = \frac{1}{\cos\theta}$$

c.

LS	RS
$\tan 60^\circ$	$\dfrac{1}{\cot 60^\circ}$
$=\dfrac{\sqrt{3}}{1}$	$=\dfrac{1}{\frac{1}{\sqrt{3}}}$
$=\sqrt{3}$	$=\sqrt{3}$

LS = RS

$$\therefore \tan\theta = \frac{1}{\cot\theta}$$

3. a. $\tan\theta\csc\theta = \dfrac{\cancel{\sin\theta}}{\cos\theta}\times\dfrac{1}{\cancel{\sin\theta}}$

$=\dfrac{1}{\cos\theta}$

$=\sec\theta$

b. $\dfrac{\tan\theta}{\sec\theta} = \dfrac{\sin\theta}{\cos\theta}\times\dfrac{1}{\sec\theta}$

$=\dfrac{\sin\theta}{\cancel{\cos\theta}}\times\cancel{\cos\theta}$

$=\sin\theta$

4. $\sec\theta\csc\theta\cot\theta = \dfrac{1}{\cancel{\cos\theta}}\times\dfrac{1}{\sin\theta}\times\dfrac{\cancel{\cos\theta}}{\sin\theta}$

$=\dfrac{1}{\sin^2\theta}$

5. a. 1 2 0 sin x^2 + 1 2 0 cos x^2 =

1.

$\therefore \sin^2\theta + \cos^2\theta = 1$

b. 1 2 0 cos $\frac{1}{x}$ x^2 − 1 2 0 tan x^2 =

1.

$\therefore \sec^2\theta - \tan^2\theta = 1$

c. 1 2 0 tan × 1 2 0 tan $\frac{1}{x}$ =

1.

$\therefore \tan\theta\cot\theta = 1$

6. a. False b. True c. True

7. Use the Pythagorean Theorem.

$$x^2 + y^2 = r^2$$

$$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2} \qquad \left(\text{Divide by } x^2.\right)$$

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \left(\frac{y}{x} = \tan\theta \text{ and } \frac{r}{x} = \sec\theta\right)$$

b. $\sec\theta - \cos\theta = \frac{1}{\cos\theta} - \cos\theta$

$$= \frac{1-\cos^2\theta}{\cos\theta} \text{ or } \frac{\sin^2\theta}{\cos\theta}$$

8. **a.** $\sin^2\theta = 1 - \cos^2\theta$

b. $\sec^2\theta = \frac{1}{\cos^2\theta}$

c. $\tan\theta \sin\theta = \left(\frac{\sin\theta}{\cos\theta}\right)(\sin\theta)$

$$= \frac{\sin^2\theta}{\cos\theta}$$

$$= \frac{1-\cos^2\theta}{\cos\theta}$$

d. $\frac{\csc\theta}{\cot\theta} = \frac{\frac{1}{\sin\theta}}{\frac{\cos\theta}{\sin\theta}}$

$$= \frac{1}{\cos\theta}$$

c. $\frac{1+\cot^2\theta}{\sec^2\theta} = \frac{\csc^2\theta}{\sec^2\theta}$

$$= \frac{\frac{1}{\sin^2\theta}}{\frac{1}{\cos^2\theta}}$$

$$= \frac{\cos^2\theta}{\sin^2\theta}$$

or

$$\frac{1-\sin^2\theta}{\sin^2\theta}$$

or

$$\frac{\cos^2\theta}{1-\cos^2\theta}$$

d. $\frac{\csc\theta}{\tan\theta} = \frac{\frac{1}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}}$

$$= \frac{\cos\theta}{\sin^2\theta}$$

or

$$\frac{\cos\theta}{1-\cos^2\theta}$$

9. Answers may vary.

a. $\frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}$

$$= \sin\theta$$

e. $\cot\theta + \sec\theta = \frac{\cos\theta}{\sin\theta} + \frac{1}{\cos\theta}$

$$= \frac{\cos^2\theta + \sin\theta}{\sin\theta\cos\theta}$$

f. $\dfrac{1}{1+\cot^2\theta}=\dfrac{1}{\csc^2\theta}$

$=\dfrac{1}{\frac{1}{\sin^2\theta}}$

$=\sin^2\theta$

10. a. $\tan^2\theta-\left(1+\tan^2\theta\right)=\tan^2\theta-1-\tan^2\theta$

$=-1$

b. $\sin y\ \csc^2 y=\sin y\left(\dfrac{1}{\sin^2 y}\right)$

$=\dfrac{1}{\sin y}$

$=\csc y$

c. $\cos x+(\sin x)\left(\dfrac{\sin x}{\cos x}\right)=\dfrac{\cos^2 x+\sin^2 x}{\cos x}$

$=\dfrac{1}{\cos x}$

$=\sec x$

d. $\dfrac{\frac{1}{\cos D}}{\frac{\sin D}{\cos D}}=\dfrac{1}{\sin D}$

$=\csc D$

11. a. $\dfrac{1-\sin^2\theta}{\csc^2\theta-1}=\dfrac{\cos^2\theta}{\cot^2\theta}$

$=\cos^2\theta\left(\dfrac{1}{\cot^2\theta}\right)$

$=\cancel{\cos^2\theta}\left(\dfrac{\sin^2\theta}{\cancel{\cos^2\theta}}\right)$

$=\sin^2\theta$

Therefore, the left side is identical to the right side.

b. $\sec^2\theta-\sin^2\theta=1+\tan^2\theta-\sin^2\theta$

$=1-\sin^2\theta+\tan^2\theta$

$=\cos^2\theta+\tan^2\theta$

Therefore, the left side is identical to the right side.

c. $\sec\theta\ \csc\theta\ \cot\theta=\left(\dfrac{1}{\cancel{\cos\theta}}\right)\left(\dfrac{1}{\sin\theta}\right)\left(\dfrac{\cancel{\cos\theta}}{\sin\theta}\right)$

$=\dfrac{1}{\sin^2\theta}$

$=\csc^2\theta$

Therefore, the left side is identical to the right side.

12. **a.** $(1-\cos\theta)(1+\cos\theta)$

b. $\sin\alpha(1-\sin\alpha)$

c. $\left(\tan^2\beta-\cot^2\beta\right)\left(\tan^2\beta+\cot^2\beta\right)$

$$=(\tan\beta-\cot\beta)(\tan\beta+\cot\beta)\left(\tan^2\beta+\cot^2\beta\right)$$

d. $(2\cos y-1)(2\cos y-1)$

e. $\sec x(\sin x+1)$

f. $\sin\theta(\sec\theta-1)+(\sec\theta-1)=(\sec\theta-1)(\sin\theta+1)$

13. **a.**
$$\begin{aligned}\cos A\tan A&=\cancel{\cos A}\left(\frac{\sin A}{\cancel{\cos A}}\right)\\&=\sin A\end{aligned}$$

b.
$$\begin{aligned}\sin^3 A+\sin A\cos^2 A&=\sin A\left(\sin^2 A+\cos^2 A\right)\\&=\sin A\\&=\frac{1}{\csc A}\end{aligned}$$

c.
$$\begin{aligned}\sec y-\tan y\sin y&=\frac{1}{\cos y}-\left(\frac{\sin y}{\cos y}\right)(\sin y)\\&=\frac{1-\sin^2 y}{\cos y}\\&=\frac{\cos^2 y}{\cos y}\\&=\cos y\end{aligned}$$

d.
$$\begin{aligned}\cos^4\theta-\sin^4\theta&=\left(\cos^2\theta-\sin^2\theta\right)\left(\cos^2\theta+\sin^2\theta\right)\\&=\cos^2\theta-\sin^2\theta\\&=1-\sin^2\theta-\sin^2\theta\\&=1-2\sin^2\theta\end{aligned}$$

e.
$$\begin{aligned}\sec^2 x+\sec x\tan x&=\frac{1}{\cos^2 x}+\frac{1}{\cos x}\left(\frac{\sin x}{\cos x}\right)\\&=\frac{1+\sin x}{\cos^2 x}\\&=\frac{1+\sin x}{1-\sin^2 x}\\&=\frac{\cancel{(1+\sin x)}}{\cancel{(1+\sin x)}(1-\sin x)}\\&=\frac{1}{1-\sin x}\end{aligned}$$

f. $(1+\cot A)^2 = 1+2\cot A+\cot^2 A$

$= 1+\cot^2 A+2\cot A$

$= \csc^2 A+2\cot A$

g. $\dfrac{\tan\theta-\sin\theta}{\tan\theta\sin\theta} = \dfrac{\frac{\sin\theta}{\cos\theta}-\sin\theta}{\left(\frac{\sin\theta}{\cos\theta}\right)(\sin\theta)}$

$= \dfrac{\frac{\sin\theta-\sin\theta\cos\theta}{\cos\theta}}{\frac{\sin^2\theta}{\cos\theta}}$

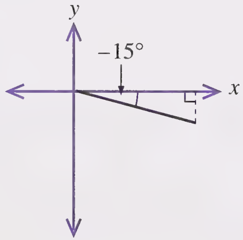
$= \dfrac{\sin\theta-\sin\theta\cos\theta}{\sin^2\theta}$

$= \dfrac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$

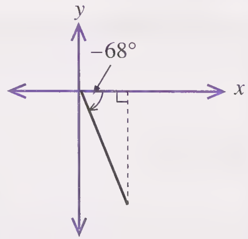
$= \dfrac{1-\cos\theta}{\sin\theta}$

Section 5: Activity 2

1. a. $\tan(-15°) = -\tan 15°$



b. $\csc(-68°) = -\csc 68°$

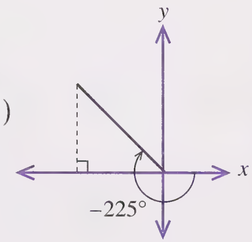


c. $\cos(-\theta) = \cos\theta$

$\therefore\ \cos(-225°) = \cos 225°$

$= \cos(180°+45°)$

$= -\cos 45°$

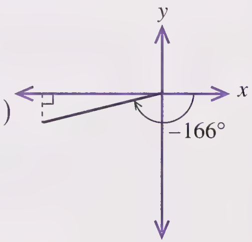


d. $\cot(-\theta) = -\cot\theta$

$\therefore\ \cot(-166°) = -\cot 166°$

$= -\cot(180°-14°)$

$= \cot 14°$



2. a. $-\dfrac{1}{2}$ **b.** -1 **c.** $\dfrac{1}{2}$ **d.** $-\dfrac{2}{\sqrt{3}}$

3. $\cos A = \frac{2}{5}$

4. $\cos A = \frac{2}{5}$

$$\begin{aligned} \sec A &= \frac{1}{\cos A} \\ &= \frac{1}{\frac{2}{5}} \\ &= \frac{5}{2} \end{aligned}$$

5. $$\begin{aligned} \sin A &= \cos 3A \\ \sin A &= \sin (90° - 3A) \\ A &= 90° - 3A \\ 4A &= 90° \\ A &= \frac{90°}{4} \\ &= 22.5° \end{aligned}$$

6. a. $$\begin{aligned} -\sin(-2\theta) &= -(-\sin 2\theta) \\ &= \sin 2\theta \end{aligned}$$

b. $$\begin{aligned} -\cot(-3-4\theta) &= -\cot[-(3+4\theta)] \\ &= -[-\cot(3+4\theta)] \\ &= \cot(3+4\theta) \end{aligned}$$

7. a. $\cos(90° - 5r) = \sin 5r$

b. $\csc\left(\frac{\pi}{2} - 44k\right) = \sec 44k$

c. $$\begin{aligned} \sin(-90° + 5m) &= \sin[-(90° - 5m)] \\ &= -\sin(90° - 5m) \\ &= -\cos 5m \end{aligned}$$

d. $\tan\left(-\frac{\pi}{2} + 7b\right) = -\cot 7b$

Section 5: Activity 3

1.

LS	RS
$\cos(a+b)$	$\cos a + \cos b$
$= \cos(30° + 120°)$	$= \cos 30° + \cos 120°$
$= \cos 150°$	$= \frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)$
$= -\frac{\sqrt{3}}{2}$	$= \frac{\sqrt{3}-1}{2}$

LS $\neq$ RS

2. a. $$\begin{aligned} \cos(\pi - \theta) &= \cos \pi \cos \theta + \sin \pi \sin \theta \\ &= (-1)\cos\theta + (0)\sin\theta \\ &= -\cos\theta \end{aligned}$$

b. $$\tan\left(\frac{\pi}{4}+\theta\right)=\frac{\sin\left(\frac{\pi}{4}+\theta\right)}{\cos\left(\frac{\pi}{4}+\theta\right)}$$

$$=\frac{\sin\frac{\pi}{4}\cos\theta+\cos\frac{\pi}{4}\sin\theta}{\cos\frac{\pi}{4}\cos\theta-\sin\frac{\pi}{4}\sin\theta}$$

$$=\frac{\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta}{\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta}$$

$$=\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$$

c. $$\sin\left(\theta-\frac{\pi}{2}\right)=\sin\theta\cos\frac{\pi}{2}-\cos\theta\sin\frac{\pi}{2}$$

$$=-\cos\theta$$

3. a.

LS	RS
$\sin(A+B)$	$\sin A\cos B+\cos A\sin B$
$=\sin(210°+120°)$	$=\sin 210°\cos 120°+\cos 210°\sin 120°$
$=\sin 330°$	$=\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)+\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$
$=-\frac{1}{2}$	$=\frac{1}{4}-\frac{3}{4}$
	$=-\frac{1}{2}$

LS = RS

LS	RS
$\cos(A-B)$	$\cos A \cos B+\sin A \sin B$
$=\cos(210°-120°)$	$=\cos 210° \cos 120°+\sin 210° \sin 120°$
$=\cos 90°$	$=\left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right)+\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$
$=0$	$=\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4}$
	$=0$

LS = RS

b.

LS	RS
$\sin\left(\frac{5\pi}{6}+\frac{\pi}{3}\right)$	$\sin\frac{5\pi}{6}\cos\frac{\pi}{3}+\cos\frac{5\pi}{6}\sin\frac{\pi}{3}$
$=\sin\frac{7\pi}{6}$	$=\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$
$=-\frac{1}{2}$	$=\frac{1}{4}-\frac{3}{4}$
	$=-\frac{1}{2}$

LS = RS

LS	RS
$\cos\left(\frac{5\pi}{6}-\frac{\pi}{3}\right)$	$\cos\frac{5\pi}{6}\cos\frac{\pi}{3}+\sin\frac{5\pi}{6}\sin\frac{\pi}{3}$
$=\cos\frac{3\pi}{6}$	$=\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$
$=0$	$=0$

LS = RS

4. **a.** $\sin(125° - 40°) = \sin 85°$

b. $\cos\frac{\pi}{6}\cos\frac{2\pi}{3} + \sin\frac{\pi}{6}\sin\frac{2\pi}{3}$

$$= \cos\left(\frac{\pi}{6} - \frac{2\pi}{3}\right)$$
$$= \cos\frac{3\pi}{6}$$
$$= \cos\frac{\pi}{2}$$

c. $\sin(5d + 2d) = \sin 7d$

5. Answers may vary. One sample is given for each question.

a. $\cos(60° + 45°) = \cos 60° \cos 45° - \sin 60° \sin 45°$

$$= \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

b. $\cos\frac{\pi}{12} = \cos 15° = \cos(45° - 30°)$

$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

6. $\sin(\beta - \theta) = \sin\beta \cos\theta - \cos\beta \sin\theta$

$$= \left(\frac{5}{13}\right)\left(\frac{3}{5}\right) - \left(\frac{12}{13}\right)\left(\frac{4}{5}\right)$$
$$= -\frac{33}{65}$$

7. $\sin(-15°) = \sin(45° - 60°)$

$$= \sin 45° \cos 60° - \cos 45° \sin 60°$$
$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}$$
$$= \frac{1 - \sqrt{3}}{2\sqrt{2}}$$
$$= \frac{1 - \sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

8. **a.**

$$\begin{aligned}\sin 285^\circ &= \sin(360^\circ - 75^\circ)\\ &= \sin(-75^\circ)\\ &= -\sin 75^\circ\\ &= -\sin(30^\circ + 45^\circ)\\ &= -(\sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ)\\ &= -\left[\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{2}}\right)\right]\\ &= -\left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\right)\\ &= -\left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)\\ &= -\left(\frac{1+\sqrt{3}}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right)\\ &= -\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)\\ &= \frac{-\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

b.

$$\begin{aligned}\cos 255^\circ &= \cos(180^\circ + 75^\circ)\\ &= -\cos 75^\circ\\ &= -\cos(45^\circ + 30^\circ)\\ &= -(\cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ)\\ &= -\left[\frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{\sqrt{2}}\left(\frac{1}{2}\right)\right]\\ &= -\left(\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}\right)\\ &= -\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\\ &= -\left(\frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right)\\ &= -\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)\\ &= \frac{-\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

9.

$$\begin{aligned}\cos 2x &= \cos(x + x)\\ &= -\frac{7}{25}\end{aligned}$$

10. **a.**

LS	RS
$\sin 2A$	$\dfrac{2\tan A}{1+\tan^2 A}$
$=\sin(A+A)$	$=\dfrac{2\tan A}{\sec^2 A}$
$=2\sin A\cos A$	$=\left(\dfrac{2\sin A}{\cos A}\right)\left(\cos^2 A\right)$
	$=2\sin A\cos A$

LS = RS

b.

LS	RS
$\csc 2x$	$\cot x-\dfrac{\cos 2x}{\sin 2x}$
$=\dfrac{1}{\sin 2x}$	$=\dfrac{\cos x}{\sin x}-\dfrac{\left(\cos^2 x-\sin^2 x\right)}{2\sin x\cos x}$
$=\dfrac{1}{2\sin x\cos x}$	$=\dfrac{\cos x}{\sin x}-\dfrac{\left(1-\sin^2 x-\sin^2 x\right)}{2\sin x\cos x}$
	$=\dfrac{2\cos^2 x-1+2\sin^2 x}{2\sin x\cos x}$
	$=\dfrac{2\left(\cos^2 x+\sin^2 x\right)-1}{2\sin x\cos x}$
	$=\dfrac{1}{2\sin x\cos x}$

LS = RS

11. **a.** True $\left(-\frac{1}{2}=-\frac{1}{2}\right)$

b. False $\left(-\frac{\sqrt{3}}{2}\neq\frac{\sqrt{3}}{2}\right)$

c. False $\left(1\neq\frac{1+\sqrt{3}}{2}\right)$

d. False $\left(\frac{1}{\sqrt{3}}\neq\frac{-1}{\sqrt{3}}\right)$

e. True $\left(\frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\right)$

12. $\sin\alpha=\frac{5}{13}$, $\cos\alpha=\frac{12}{13}$, $\sin\beta=\frac{4}{5}$, and $\cos\beta=\frac{3}{5}$

$$\begin{aligned}\sin(\alpha-\beta)&=\sin\alpha\cos\beta-\cos\alpha\sin\beta\\&=\left(\frac{5}{13}\right)\left(\frac{3}{5}\right)-\left(\frac{12}{13}\right)\left(\frac{4}{5}\right)\\&=\frac{15}{65}-\frac{48}{65}\\&=-\frac{33}{65}\end{aligned}$$

13. Since $\cos(A-B)=\cos A\cos B+\sin A\sin B$, you must determine $\cos A$ and $\sin B$.

$$\sin A=\frac{12}{13}$$

$$\sin^2 A=\frac{144}{169}$$

$$\begin{aligned}\sin^2 A+\cos^2 A&=1\\ \cos^2 A&=1-\sin^2 A\\ \cos^2 A&=1-\frac{144}{169}\\ \cos^2 A&=\frac{25}{169}\\ \cos A&=\frac{5}{13}\qquad(\text{since } A \text{ is acute})\end{aligned}$$

$$\cos B=\frac{3}{5}$$

$$\cos^2 B=\frac{9}{25}$$

$$\begin{aligned}\sin^2 B+\cos^2 B&=1\\ \sin^2 B&=1-\cos^2 B\\ \sin^2 B&=1-\frac{9}{25}\\ \sin^2 B&=\frac{16}{25}\\ \sin B&=\frac{4}{5}\qquad(\text{since } B \text{ is acute})\end{aligned}$$

$$\begin{aligned}\therefore\ \cos(A-B)&=\cos A\cos B+\sin A\sin B\\ &=\left(\frac{5}{13}\right)\left(\frac{3}{5}\right)+\left(\frac{12}{13}\right)\left(\frac{4}{5}\right)\\ &=\frac{15}{65}+\frac{48}{65}\\ &=\frac{63}{65}\end{aligned}$$

Section 5: Follow-up Activities

Extra Help

1. **a.** positive **b.** negative
   c. positive **d.** negative

2. **a.** $\cot(-100°)$ is in the third quadrant; therefore, it is positive.

$$\begin{aligned}\cot(-100°)&=\cot(180°-100°)\\ &=\cot 80°\end{aligned}$$

b. $\cos(-269°)$ is in the second quadrant; therefore, it is negative.

$$\begin{aligned}\cos(-269°)&=\cos(360°-269°)\\ &=\cos(91°)\\ &=\cos(180°-91°)\\ &=-\cos 89°\end{aligned}$$

c. $\tan(-230°)$ is in the second quadrant; therefore, it is negative.

$$\begin{aligned}\tan(-230°) &= \tan(360° - 230°)\\ &= \tan 130°\\ &= \tan(180° - 130°)\\ &= -\tan 50°\end{aligned}$$

d. $\sin(-150°)$ is in the third quadrant; therefore, it is negative.

$$\begin{aligned}\sin(-150°) &= \sin(180° - 150°)\\ &= -\sin 30°\end{aligned}$$

3.
$$\begin{aligned}\cos(\alpha + \beta) &= \cos(180° + \theta)\\ &= \cos 180°(\cos\theta) - \sin 180°(\sin\theta)\\ &= -\cos\theta - 0\sin\theta\\ &= -\cos\theta\end{aligned}$$

4.
$$\begin{aligned}\sin(A - B) &= \sin(180° - \theta)\\ &= \sin 180°\cos\theta - \cos 180°\sin\theta\\ &= 0(\cos\theta) - (-1)\sin\theta\\ &= \sin\theta\end{aligned}$$

Enrichment

1. a.

LS	RS
$\dfrac{\tan x}{\sec x - \tan x}$	$\sec^2 x + \sec x \tan x - 1$
$= \dfrac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}}$	$= \tan^2 x + \sec x \tan x$
$= \dfrac{\sin x}{1 - \sin x} \times \dfrac{1 + \sin x}{1 + \sin x}$	$= \dfrac{\sin^2 x}{\cos^2 x} + \left(\dfrac{1}{\cos x}\right)\left(\dfrac{\sin x}{\cos x}\right)$
$= \dfrac{\sin^2 x + \sin x}{1 - \sin^2 x}$	$= \dfrac{\sin^2 x + \sin x}{\cos^2 x}$
	$= \dfrac{\sin^2 x + \sin x}{1 - \sin^2 x}$

LS = RS

b.

LS	RS
$\sec x \cos x$	$\tan x \cot x$
$= \left(\dfrac{1}{\cos x}\right)(\cos x)$	$= \left(\dfrac{1}{\cot x}\right)(\cot x)$
$= 1$	$= 1$

LS = RS

c.

LS	RS
$\dfrac{\tan^2\theta+1}{\cot^2\theta+1}$	$\tan^2\theta$
$=\dfrac{\sec^2\theta}{\csc^2\theta}$	$=\dfrac{\sin^2\theta}{\cos^2\theta}$
$=\dfrac{\frac{1}{\cos^2\theta}}{\frac{1}{\sin^2\theta}}$	
$=\dfrac{\sin^2\theta}{\cos^2\theta}$	

LS = RS

d.

LS	RS
$\dfrac{\cos 2x}{1+\sin 2x}$	$\dfrac{\cot x-1}{\cot x+1}$
$=\dfrac{\cos x\cos x-\sin x\sin x}{1+\sin x\cos x+\cos x\sin x}$	$=\dfrac{\frac{\cos x}{\sin x}-1}{\frac{\cos x}{\sin x}+1}$
$=\dfrac{\cos^2 x-\sin^2 x}{1+2\sin x\cos x}$	$=\dfrac{\frac{\cos x-\sin x}{\sin x}}{\frac{\cos x+\sin x}{\sin x}}$
$=\dfrac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^2 x+2\sin x\cos x}$	$=\dfrac{\cos x-\sin x}{\cos x+\sin x}$
$=\dfrac{(\cos x+\sin x)(\cos x-\sin x)}{(\cos x+\sin x)^2}$	
$=\dfrac{\cos x-\sin x}{\cos x+\sin x}$	

LS = RS

2. a. $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$$\begin{aligned}\therefore \frac{\tan m-\tan 6m}{1+\tan m\tan 6m} &= \tan(m-6m)\\ &= \tan(-5m)\\ &= -\tan 5m\end{aligned}$$

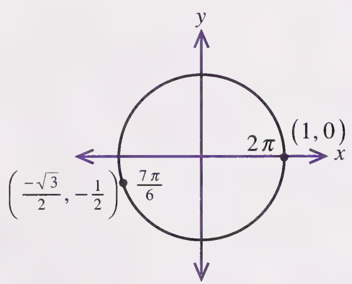
b. $\cos(A+B)=\cos A\cos B-\sin A\sin B$

$$\begin{aligned}\therefore \cos 7x\cos 2x-\sin 7x\sin 2x &= \cos(7x+2x)\\ &= \cos 9x\end{aligned}$$

3. $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

$$\begin{aligned}\therefore \tan\left(2\pi+\frac{7\pi}{6}\right) &= \frac{\tan 2\pi+\tan\frac{7\pi}{6}}{1-\tan 2\pi\tan\frac{7\pi}{6}}\\ &= \frac{0+\frac{1}{\sqrt{3}}}{1-0\left(\frac{1}{\sqrt{3}}\right)}\\ &= \frac{1}{\sqrt{3}}\\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

Divide $\frac{19\pi}{6}$ into two convenient values.



5.
$$\begin{aligned}\tan 2A &= \frac{\sin(A+A)}{\cos(A+A)}\\ &= \frac{2\sin A \cos A}{\cos^2 A - \sin^2 A}\\ &= \frac{2\frac{\sin A}{\cos A}}{1-\frac{\sin^2 A}{\cos^2 A}} \qquad \text{(Divide the numerator and denominator by } \cos^2 A.\text{)}\\ &= \frac{2\tan A}{1-\tan^2 A}\end{aligned}$$

6.
$$\begin{aligned}\cos 3x &= \cos(2x+x)\\ &= \cos 2x \cos x - \sin 2x \sin x\\ &= \left(\cos^2 x - \sin^2 x\right)\cos x - (2\sin x \cos x)\sin x\\ &= \cos^3 x - \cos x \sin^2 x - 2\sin^2 x \cos x\\ &= \cos^3 x - 3\sin^2 x \cos x\end{aligned}$$

4.
$$\begin{aligned}&\cos(x+y)\cos(x-y)\\ &= (\cos x \cos y - \sin x \sin y)(\cos x \cos y + \sin x \sin y)\\ &= \cos^2 x \cos^2 y - \sin^2 x \sin^2 y\\ &= \cos^2 x\left(1-\sin^2 y\right) - \sin^2 y\left(1-\cos^2 x\right)\\ &= \cos^2 x - \cos^2 x \sin^2 y - \sin^2 y + \sin^2 y \cos^2 x\\ &= \cos^2 x - \sin^2 y\\ &= \cos^2 x + \cos^2 y - 1\end{aligned}$$

7.

LS	RS
$\sin(A+B)$	$\sin A \cos B + \cos A \sin B$
$= \sin\left(\frac{4\pi}{3} - \frac{5\pi}{6}\right)$	$= \sin\frac{4\pi}{3}\cos\left(-\frac{5\pi}{6}\right) + \cos\frac{4\pi}{3}\sin\left(-\frac{5\pi}{6}\right)$
$= \sin\frac{3\pi}{6}$	$= \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)$
$= 1$	$= 1$

LS = RS

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